

A 13 stage combined order 7 and 8 Runge-Kutta scheme of Tsitouras and Papakostas

See: Cheap Error Estimation for Runge-Kutta methods, by Ch. Tsitouras and S.N. Papakostas,
Siam Journal on Scientific Computing, Vol. 20, Issue 6, Nov 1999.

The nodes of the scheme are:

$$c_2 = \frac{9}{142}, c_3 = \frac{24514}{238491}, c_4 = \frac{12257}{79497}, c_5 = \frac{50}{129}, c_6 = \frac{34}{73}, c_7 = \frac{23}{148}, c_8 = \frac{142}{141}, c_9 = \frac{29104120198}{33218531707}, c_{10} = \frac{83}{91}, c_{11} = \frac{143}{149}, c_{12} = 1, \\ c_{13} = 1.$$

Note: The node $c_8 = \frac{142}{141}$ of this scheme lies outside the interval $[0, 1]$. This means that the scheme cannot be used in a situation where a desired numerical solution has a right hand boundary caused, for example, by the slope field being undefined.

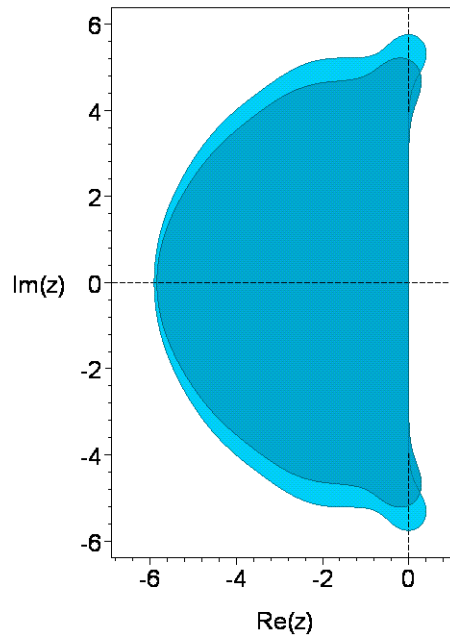
The principal error norm, that is, the 2-norm of the principal error terms is: $0.7348655385 \times 10^{(-6)}$.

The principal error norm of the order 7 embedded scheme is: $0.1056537628 \times 10^{(-4)}$.

The maximum magnitude of the linking coefficients is: 11.72351530.

The 2-norm of the linking coefficients is: 36.21710741.

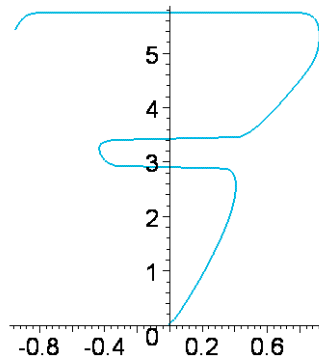
The stability regions for the two schemes are shown in the following picture.



The stability region of the order 7 scheme appears in the darker shade.

The real stability intervals of the order 8 and 7 schemes are respectively $[-5.9069, 0]$ and $[-5.8443, 0]$.

The following picture shows the result of distorting the boundary curve of the stability region of the order 8 scheme horizontally by taking the 11th root of the real part of points along the curve.



The stability region intersects the nonnegative imaginary axis in the union of intervals: $[0, 2.9149] \cup [3.4144, 5.7536]$.

The coefficients in exact form are:

$c[2]=9/142,$
 $c[3]=24514/238491,$
 $c[4]=12257/79497,$
 $c[5]=50/129,$
 $c[6]=34/73,$
 $c[7]=23/148,$
 $c[8]=142/141,$
 $c[9]=29104120198/33218531707,$
 $c[10]=83/91,$
 $c[11]=143/149,$
 $c[12]=1,$
 $c[13]=1,$

$a[2,1]=9/142,$
 $a[3,1]=9950845450/511901613729,$
 $a[3,2]=42666469916/511901613729,$
 $a[4,1]=12257/317988,$
 $a[4,2]=0,$
 $a[4,3]=12257/105996,$
 $a[5,1]=14131686489425/35833975601529,$
 $a[5,2]=0,$
 $a[5,3]=-17700454220625/11944658533843,$
 $a[5,4]=17619604667500/11944658533843,$
 $a[6,1]=4418011/96055225,$
 $a[6,2]=0,$
 $a[6,3]=0,$
 $a[6,4]=9757757988/41995818067,$
 $a[6,5]=19921512441/106300094275,$
 $a[7,1]=480285555889619/7997789253816320,$
 $a[7,2]=0,$
 $a[7,3]=0,$
 $a[7,4]=52162106556696460538643/464887033284472899365888,$
 $a[7,5]=-16450769949384489/490009784398315520,$
 $a[7,6]=864813381633887/51718297665732608,$
 $a[8,1]=-7001048352088587137143/4449830351030385148800,$
 $a[8,2]=0,$
 $a[8,3]=0,$
 $a[8,4]=-2522765548294599044197935933/1915963195493758447677901792,$
 $a[8,5]=-318556182235222634647116091/27172411532484037214054400,$
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 $a[8,7]=254/39,$
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 $a[9,2]=0,$
 $a[9,3]=0,$
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 $498517112641608872807821838566847987729514312398278633788185835348997643218553476,$
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a[11,2]=0,
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a[11,6]=26823469063654084387375587616552322383082061411417182757389742951/
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a[11,8]=64747617454909275289531520412519442831235890581/
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a[11,10]=-10612202518573994431153697720606405883/67082546658259846778754594976831647575,
a[12,1]=-7534081165544982478296202335922049210803875045423/
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b[1]=55038446513529253801/1239280570055853383520,
b[2]=0,
b[3]=0,
b[4]=0,
b[5]=0,
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b[8]=-4237087214169934312729607487/12735791394214625116604076160,
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b[10]=-1108225296327029096435947/405679075893979729103310,
b[11]=2460988291206213825688467985/1756342789520947764222671739,
b[12]=4808707937311/50545545388065,
b[13]=0,

b*[1]=715953338020208413/16094552857868225760,
b*[2]=0,
b*[3]=0,
b*[4]=0,
b*[5]=0,
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b*[7]=184309146777472302831695872/742417767522160213324368465,
b*[8]=-509771598215811385123057257/210282269969085698264101760,
b*[9]=2701602489646143640362891402924962500766379885231830470264478480621389/
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b*[11]=0,
b*[12]=0,
b*[13]=5752173075461/1925544586212.