

An 11 stage, order 7 Runge-Kutta scheme with an order 6 embedded scheme

See: Explicit Runge-Kutta Pairs with One More Derivative Evaluation than the Minimum, by P.W.Sharp and E.Smart, Siam Journal of Scientific Computing, Vol. 14, No. 2, pages. 338-348, March 1993.

The nodes of the scheme are:

$$c_2 = \frac{1}{24}, c_3 = \frac{897}{1072}, c_4 = \frac{160}{499}, c_5 = \frac{141}{763}, c_6 = \frac{169}{192}, c_7 = \frac{998622639}{2007664595}, c_8 = \frac{104}{105}, c_9 = \frac{20}{21}, c_{10} = 1, c_{11} = 1.$$

The principal error norm of the order 7 scheme, that is, the 2-norm of the principal error terms is: $0.3751159742 \times 10^{(-4)}$.

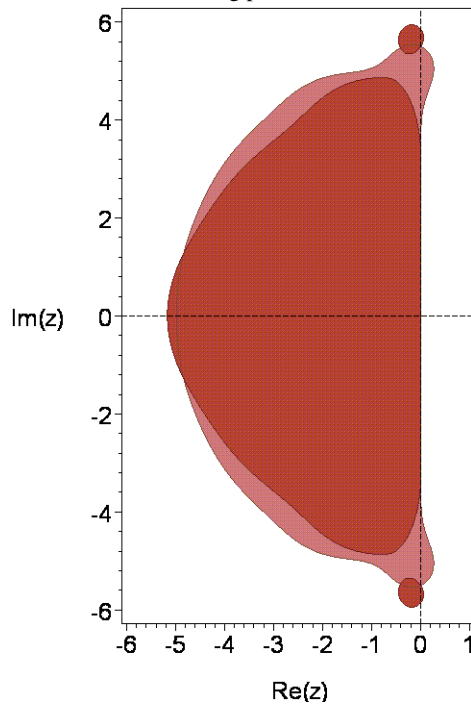
The 2-norm of the order 9 error terms is $0.1877540811 \times 10^{(-3)}$, which is approximately 5.005 times the principal error norm.

The principal error norm of the order 6 embedded scheme is: $0.5848132143 \times 10^{(-4)}$.

The maximum magnitude of the linking coefficients is: 8.401881683.

The 2-norm of the linking coefficients is: 19.76223644.

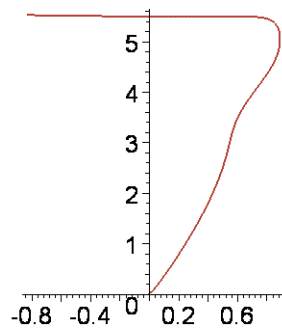
The stability regions for the two schemes are shown in the following picture.



The stability region of the order 6 scheme appears in the darker shade.

The real stability intervals of the order 7 and 6 schemes are respectively $[-4.9826, 0]$ and $[-5.1690, 0]$.

The following picture shows the result of distorting the boundary curve of the stability region of the order 7 scheme horizontally by taking the 11th root of the real part of points along the curve.



The stability region intersects the nonnegative imaginary axis in the interval: $[0, 5.5047]$.

The coefficients in exact form are:

- $c[2]=1/24,$
- $c[3]=897/1072,$
- $c[4]=160/499,$
- $c[5]=141/763,$
- $c[6]=169/192,$
- $c[7]=998622639/2007664595,$
- $c[8]=104/105,$
- $c[9]=20/21,$
- $c[10]=1,$
- $c[11]=1,$

- $a[2,1]=1/24,$
- $a[3,1]=-2173431/287296,$
- $a[3,2]=2413827/287296,$
- $a[4,1]=57894880/223353897,$
- $a[4,2]=0,$
- $a[4,3]=13721600/223353897,$
- $a[5,1]=5056664940861/42500572528960,$
- $a[5,2]=0,$
- $a[5,3]=-286249149686016/36667767392227699,$
- $a[5,4]=2889259817673483/39243095536832320,$
- $a[6,1]=174/407,$
- $a[6,2]=0,$
- $a[6,3]=861765466269918263/4970009789268581376,$
- $a[6,4]=10826192777269994995/7712691245665173504,$
- $a[6,5]=-16750236560040896731/14897194035750715392,$
- $a[7,1]=-3439778790612913527970381936467065474966136664036779/$
 $39381209389424561778044106217782104388290028625000000,$
- $a[7,2]=0,$
- $a[7,3]=2631414486943314085034368959766533330714638432739014413056/$
 $42832809612378095870766530128571820142439373914153488671875,$
- $a[7,4]=-3226817813200049695169462352322166717564113587934800370011/$
 $40015346463976160761929086193516183832403838724337375000000,$
- $a[7,5]=786190577550706940215874040419538587608361105183350570182/$
 $1261335096542994096899051116924589799398264209359351953125,$
- $a[7,6]=-5411006332774526240948446773664264724938752/279717068193641240266628706086067991948046875,$
- $a[8,1]=15322017632690457879761200795014874391/29304923702961208406725757182921875000,$
- $a[8,2]=0,$
- $a[8,3]=3346824815910846126629531019335170048/2298048540174087438681694595766796875,$
- $a[8,4]=189748010561998005455394521375650921/29817877073894091150172096921875000,$
- $a[8,5]=-382845331583835193195089397869250404738354182/110225463285588517977748940118009136362890625,$
- $a[8,6]=-120391896911431755652090822656/233667703961223196028992578125,$
- $a[8,7]=-79695128060562505037238941490242415433886906/23692035580167289837348760854832233004866581,$
- $a[9,1]=1032897956298204654305923433045759171807509/2004961136769885546067510633666861543410000,$
- $a[9,2]=0,$
- $a[9,3]=8472900696201518374537737273138176/9559881927124203744915849518389875,$
- $a[9,4]=1046253623595072271778442420598129/248084737254798838369431846390000,$
- $a[9,5]=-2255808953526127614059281040689393645809880661461199/$
 $891519231565352783145488093784087901162293552521625,$
- $a[9,6]=-1788698788744957883891433528082038784/5923143676314255446643283231146667875,$
- $a[9,7]=-574800468754997103944423838666252602754508075193305936911875/$
 $311545184693907215706597063235165448890282539301416442871559,$
- $a[9,8]=88873364366227887565000/8176290045932637014386689,$
- $a[10,1]=86280208734474390054682197174277603231/163235672733343926284895054973116600000,$
- $a[10,2]=0,$
- $a[10,3]=871191425901232070238397371934719232/533362446801744790377544382297593125,$
- $a[10,4]=8964183417443891431743857714046506701/1273379282362081167514274901917400000,$

a[10,5]=-252047840384081563219583820196041648831131662918986674/
66729954558238813530334990145293701430108220259249375,
a[10,6]=-5290899117464226058433745200964698112/8875951095645670561315492876310816875,
a[10,7]=-306989390668037647170793389323371972068207220184067211641263783356000/
80013418079725292844034374434224082313958525560207876667654196707053,
a[10,8]=-7376114064684719178734872354875/359022955840383035894248406127836,
a[10,9]=255482840375593769438967652731/8853429096639550069183960833070,
a[11,1]=2429287816990748646637471052716009978490201/4607015764006388556345602665417429396800000,
a[11,2]=0,
a[11,3]=160046766004414315742455765106336/98065952173285752701328766250625,
a[11,4]=572651535981050835474886343012483/81435870991749105781564929600000,
a[11,5]=-1900266684289629480230017826509794378842064985811946527/
5038998773730568797417283222822299497058344239378524000,
a[11,6]=-73608385714443424401508261029604294656/123768183621850031611831870691647493125,
a[11,7]=-10527833851533262864999371634499752441444282441429608616850996735536625/
2746398581449952235460650199594860340361817126809954433857815662665184,
a[11,8]=-5616863065079321503840875/276176908218169072485950384,
a[11,9]=40097947059612729/1419769450975237280,
a[11,10]=0,

b[1]=164949402359214517/2969763122318428800,
b[2]=0,
b[3]=0,
b[4]=0,
b[5]=166164647343343629239462104853/594150218553182839761586950000,
b[6]=883515736589998531682304/6999851448505791234978125,
b[7]=110567843372870117205263375074449005180061856568412069340896875/
340717754649392931183331876846338804668211857149398834541328544,
b[8]=758177409913942921875/78785859799058312576,
b[9]=-2973333145929506703/747247079460651200,
b[10]=-1895689785077201/288707084450720,
b[11]=0,

b*[1]=20506743773621213/371220390289803600,
b*[2]=0,
b*[3]=0,
b*[4]=0,
b*[5]=268231112188987944945827753/955225431757528681288725000,
b*[6]=415490030471198546067456/304341367326338749346875,
b*[7]=108419211385790972003069732530101032173894775381027875/
337664606137936380500051156293384895353362152117883624,
b*[8]=54584684081980124625/4924116237441144536,
b*[9]=-842056888251867309/186811769865162800,
b*[10]=0,
b*[11]=-38/5.