

A modification of Hiroshi Ono's order 12 Runge-Kutta scheme with an order 9 embedded scheme

See: On the 25 stage 12th order explicit Runge-Kutta method, by Hiroshi Ono.

Transactions of the Japan Society for Industrial and applied Mathematics, Vol. 6, No. 3, (2006) pages 177 to 186.

Note: The embedded order 9 scheme requires 4 stages to be added to the 25 stage order 12 scheme.

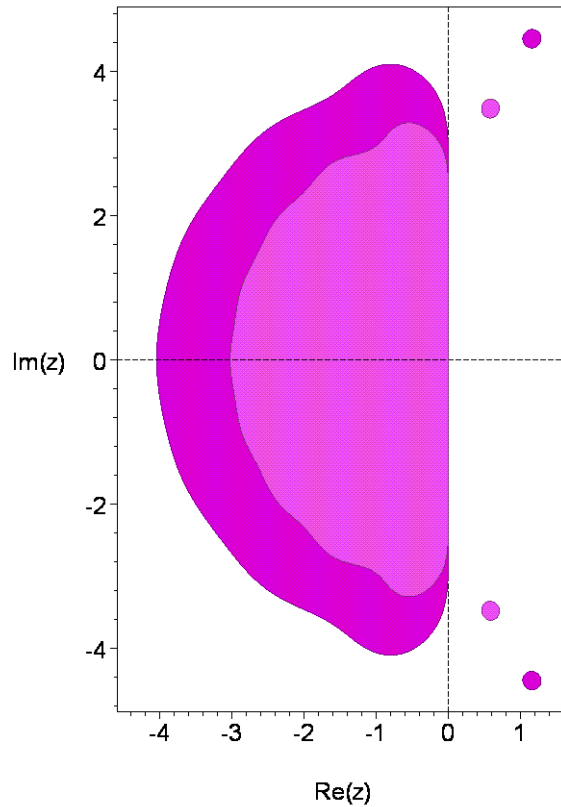
The principal error norm of the order 12 scheme, that is, the 2-norm of the principal error terms is: $0.3152572305 \times 10^{(-7)}$.

The principal error norm of the embedded order 9 scheme is: $0.7348313900 \times 10^{(-5)}$.

The maximum magnitude of the linking coefficients is 212.1164197.

The 2-norm of the linking coefficients is 384.3703602.

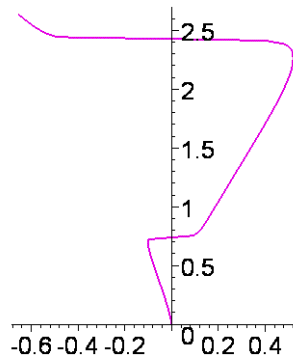
The stability regions for the two schemes are shown in the following picture.



The stability region of the order 9 scheme appears in the darker shade.

The real stability intervals of the order 12 and 9 schemes are respectively $[-3.0248, 0]$ and $[-4.0456, 0]$.

The following picture shows the result of distorting the boundary curve of the stability region of the order 12 scheme horizontally by taking the 11th root of the real part of points along the curve.



The stability region intersects the nonnegative imaginary axis in the interval $[0.7481, 2.4158]$.

a[9,1]=.1888176809184108413680419706237505704281864015973418413153299364663524364341955734913e-1,
a[9,2]=0.,
a[9,3]=0.,
a[9,4]=0.,
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a[9,6]=.8381198329740939383758467268684128106638290469173528975872944409244863365387981657651e-1,
a[9,7]=.9031585241436450659339626877786556727134244569192456020771509670971435444573825121328e-5,
a[9,8]=.9729721702550808557495179062390587533407132090396133365371679075124515126725605224924e-1,
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a[10,2]=0.,
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a[10,4]=0.,
a[10,5]=0.,
a[10,6]=-.6005539308646711394173617417282635784603029781694346062080009050806164919590318269156,
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a[11,4]=0.,
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a[14,6]=0.,
a[14,7]=0.,
a[14,8]=0.,

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a[15,7]=0.,
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a[18,3]=0.,
a[18,4]=0.,
a[18,5]=0.,
a[18,6]=0.,
a[18,7]=1.975263319684766813850955500728188106557880392559127720135666889111189136253813975729,
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a[26,19]=0.,
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a[27,22]=0.,
a[27,23]=0.,
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 a[28,27]=-.1603789707964253713820928925063521366431305782887091520824325014403564569409562398070,
 a[29,1]=46.12864603958015905056850990838704062569763496465412763519385202873225169326091596022,
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 a[29,7]=16.1136268986245124099097528833948400039234961533373225370151713993919933068137837655,
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 a[29,11]=-1.242830487244052672528847080627776989497284066852925470069357906835696725179561610962,
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 a[29,27]=-2.069534982695615656321598541301254038890059866116256078922522295738656144906973495868,
 a[29,28]=2.046912691678016537956965912259391642243284658827565955103431482290288593093460219351,

b[1]=.2380952380952380952380952380952380952380952380952380952380952380952380952380952380952e-1,
 b[2]=.11,
 b[3]=-0.17,
 b[4]=0.,
 b[5]=0.,
 b[6]=-0.19,
 b[7]=-0.21,
 b[8]=0.,
 b[9]=-0.23,
 b[10]=-0.27,
 b[11]=-0.29,
 b[12]=0.,
 b[13]=.1384130236807829740053502031450331467488136400899412345912671194817223119377730668077,
 b[14]=.2158726906049313117089355111406811389654720741957730511230185948039919737765126474781,
 b[15]=.2438095238095238095238095238095238095238095238095238095238095238095238095238095238095,
 b[16]=.2158726906049313117089355111406811389654720741957730511230185948039919737765126474781,
 b[17]=.1384130236807829740053502031450331467488136400899412345912671194817223119377730668077,
 b[18]=.29,
 b[19]=.27,
 b[20]=.23,
 b[21]=.21,
 b[22]=.19,
 b[23]=.17,
 b[24]=.11,
 b[25]=.2380952380952380952380952380952380952380952380952380952380952380952380952380952e-1,
 b[26]=0.,
 b[27]=0.,

b[28]=0.,
b[29]=0.,

b*[1]=.1357267366422036691624508570375039921213961405197383540990729271585608575744367221422e-1,
b*[2]=0.,
b*[3]=0.,
b*[4]=0.,
b*[5]=0.,
b*[6]=0.,
b*[7]=0.,
b*[8]=0.,
b*[9]=.1957242608025905233613155193355413222756712243739956539857744660017249157522413539338,
b*[10]=.6188866347435608661616060238380345321740579185698262925072441030810461130118224626650e-1,
b*[11]=.2356461254963383884566009189531765712282459575681722654363888116993755654359289982043,
b*[12]=.9356981277656948171659125355126822887878370575133833270918675400840988929847712528189e-1,
b*[13]=0.,
b*[14]=0.,
b*[15]=0.,
b*[16]=0.,
b*[17]=0.,
b*[18]=0.,
b*[19]=0.,
b*[20]=0.,
b*[21]=0.,
b*[22]=0.,
b*[23]=0.,
b*[24]=0.,
b*[25]=0.,
b*[26]=.2788382624223597882496809755901865993371492355322230983832374605891024297949723722505,
b*[27]=.4265887719284871852244002213010235951116369449606049493109912099005098392090245967637,
b*[28]=-.2878025166474501962999477241233861487151201542084751225511851437798538088192303308430,
b*[29]=-.1802605391747162424104685269536402054591231988681564193502526144322952773004003407203e-1.