Construction of a 29 stage combined order 9 and 12 Runge-Kutta scheme

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A method for the constuction of combined 9 and 12 Runge-Kutta schemes is described here. The embedded order 9 scheme is provided with the aim of it being used for error control in the standard manner. The method of construction of the order 12 scheme is motivated by a scheme of Hiroshi Ono.

See: On the 25 stage 12th order explicit Runge-Kutta method, by Hiroshi Ono.
Transactions of the Japan Society for Industrial and applied Mathematics, Vol. 6, No. 3, (2006) pages 177 to 186.

We first indicate how to construct a 25 stage order 12 Runge-Kutta scheme. We make use of various symmetry relations involving the coefficients.

Step 1.
We specify that $b_4 = 0,\ b_5 = 0,\ b_8 = 0,\ b_{12} = 0,\ c_{25} = 1$.
The symmetry relations

$$c_2 = c_{24},\ c_3 = c_{23},\ c_6 = c_{22},\ c_7 = c_{21},\ c_9 = c_{20},\ c_{10} = c_{19},\ c_{11} = c_{18},$$
$$b_2 = -b_{24},\ b_3 = -b_{23},\ b_6 = -b_{22},\ b_7 = -b_{21},\ b_9 = -b_{20},\ b_{10} = -b_{19},\ b_{11} = -b_{18}$$

provide a simplification of the quadrature conditions

$$\sum_{i=1}^{25} b_i = 1, \quad \sum_{i=1}^{25} b_i\, c_i^{\,k} = \frac{1}{k+1}, \quad k = 1\,..\,11,$$

due to a cancellation of terms.
We can solve the resulting system of equations by introducing the additional conditions $c_{13} + c_{17} = 1,\ c_{14} + c_{16} = 1,\ b_1 = b_{25},\ b_{13} = b_{17}$ and $b_{14} = b_{16}$.
Selecting the solution with $c_{13} < c_{14} < c_{16} < c_{17}$ gives

$$c_{13} = \frac{1}{2} - \frac{\sqrt{495 + 66\sqrt{15}}}{66},\ c_{14} = \frac{1}{2} - \frac{\sqrt{495 - 66\sqrt{15}}}{66},\ c_{15} = \frac{1}{2},\ c_{16} = \frac{1}{2} + \frac{\sqrt{495 - 66\sqrt{15}}}{66},\ c_{17} = \frac{1}{2} + \frac{\sqrt{495 + 66\sqrt{15}}}{66}.$$

These nodes are the zeros of the derivative $P'_6(x) = \frac{d}{dx} P_6(x)$ of the Legendre polynomial $P_6(x)$ mapped linearly from the interval $[-1, 1]$ to the interval $[0, 1]$. They provide nodes for Gauss-Lobatto integration on the interval $[0, 1]$.
We also obtain the weights

$$b_1 = \frac{1}{42},\ b_{13} = \frac{31}{175} - \frac{\sqrt{15}}{100},\ b_{14} = \frac{31}{175} + \frac{\sqrt{15}}{100},\ b_{15} = \frac{128}{525},\ \ b_{16} = \frac{31}{175} + \frac{\sqrt{15}}{100},\ b_{17} = \frac{31}{175} - \frac{\sqrt{15}}{100},\ b_{25} = \frac{1}{42}.$$

Step 2:
The stage-orders of stages 3 to 25 of the scheme are given as follows.

stage	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
stage-order	2	3	3	4	4	4	5	5	5	5	6	6	6	6	6	5	5	5	1	1	1	1	6

Thus we have the following stage-order conditions involving the nodes c_i and linking coefficients $a_{i,j}$.

$$\sum_{j=1}^{i-1} a_{i,j} = c_i \ \text{ for } i = 3\,..\,25, \qquad \sum_{j=2}^{i-1} a_{i,j}\, c_j = \frac{c_i^{\,2}}{2} \ \text{ for } i = 3\,..\,20, 25, \qquad \sum_{j=2}^{i-1} a_{i,j}\, c_j^{\,2} = \frac{c_i^{\,3}}{3} \ \text{ for } i = 4\,..\,20, 25, \qquad \sum_{j=2}^{i-1} a_{i,j}\, c_j^{\,3} = \frac{c_i^{\,4}}{4} \ \text{ for } i = 6\,..\,20, 25,$$

$$\sum_{j=2}^{i-1} a_{i,j}\, c_j^{\,4} = \frac{c_i^{\,5}}{5} \ \text{ for } i = 9\,..\,20, 25 \qquad \sum_{j=2}^{i-1} a_{i,j}\, c_j^{\,5} = \frac{c_i^{\,6}}{6} \ \text{ for } i = 13\,..\,17, 25.$$

The zero linking coefficients in the first 13 stages are indicated by the following tableau.

$$\begin{bmatrix} c_2 & a_{2,1} \\ c_3 & a_{3,1} & a_{3,2} \\ c_4 & a_{4,1} & 0 & a_{4,3} \\ c_5 & a_{5,1} & 0 & a_{5,3} & a_{5,4} \\ c_6 & a_{6,1} & 0 & 0 & a_{6,4} & a_{6,5} \\ c_7 & a_{7,1} & 0 & 0 & a_{7,4} & a_{7,5} & a_{7,6} \\ c_8 & a_{8,1} & 0 & 0 & 0 & a_{8,5} & a_{8,6} & a_{8,7} \\ c_9 & a_{9,1} & 0 & 0 & 0 & 0 & a_{9,6} & a_{9,7} & a_{9,8} \\ c_{10} & a_{10,1} & 0 & 0 & 0 & 0 & a_{10,6} & a_{10,7} & a_{10,8} & a_{10,9} \\ c_{11} & a_{11,1} & 0 & 0 & 0 & 0 & 0 & a_{11,7} & a_{11,8} & a_{11,9} & a_{11,10} \\ c_{12} & a_{12,1} & 0 & 0 & 0 & 0 & a_{12,6} & a_{12,7} & a_{12,8} & a_{12,9} & a_{12,10} & a_{12,11} \\ c_{13} & a_{13,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{13,9} & a_{13,10} & a_{13,11} & a_{13,12} \end{bmatrix}$$

Hiroshi Ono's original scheme also has $a_{12,6} = 0$ and $a_{12,7} = 0$.
We can use this information to obtain certain relations involving the nodes c_3 to c_{13}.

$$\int_0^{c_4} x\,(x - c_3)\,dx = 0 \quad \text{or} \quad c_3 = \frac{2}{3}\,c_4 \quad \text{------------- (i)}$$

$$\int_0^{c_6} x\,(x - c_4)\,(x - c_5)\,dx = 0 \quad \text{or} \quad c_5 = \frac{c_6\,(3\,c_6 - 4\,c_4)}{2\,(2\,c_6 - 3\,c_4)} \quad \text{------------- (ii)}$$

$$\int_0^{c_9} x\,(x - c_6)\,(x - c_7)\,(x - c_8)\,dx = 0 \quad \text{or} \quad c_8 = \frac{(20\,c_6\,c_7 - 15\,c_9\,c_7 + 12\,c_9^{\,2} - 15\,c_9\,c_6)\,c_9}{5\,(6\,c_6\,c_7 - 4\,c_9\,c_6 + 3\,c_9^{\,2} - 4\,c_9\,c_7)} \quad \text{------------- (iii)}$$

$$\int_0^{c_{13}} x\,(x - c_9)\,(x - c_{10})\,(x - c_{11})\,(x - c_{12})\,dx = 0 \quad \text{or}$$

$$c_{12} = \frac{(15\,c_9\,c_{13}\,c_{10} - 20\,c_9\,c_{10}\,c_{11} - 12\,c_9\,c_{13}^{\,2} + 15\,c_9\,c_{13}\,c_{11} - 12\,c_{10}\,c_{13}^{\,2} + 15\,c_{10}\,c_{13}\,c_{11} + 10\,c_{13}^{\,3} - 12\,c_{11}\,c_{13}^{\,2})\,c_{13}}{20\,c_9\,c_{13}\,c_{10} - 30\,c_9\,c_{10}\,c_{11} - 15\,c_9\,c_{13}^{\,2} + 20\,c_9\,c_{13}\,c_{11} - 15\,c_{10}\,c_{13}^{\,2} + 20\,c_{10}\,c_{13}\,c_{11} + 12\,c_{13}^{\,3} - 15\,c_{11}\,c_{13}^{\,2}} \quad \text{------------- (iv).}$$

We assume the following symmetry conditions involving the nodes c_i and weghts b_i.

$$c_2 = c_{24}, \quad c_3 = c_{23}, \quad c_6 = c_{22}, \quad c_7 = c_{21}, \quad c_9 = c_{20}, \quad c_{10} = c_{19}, \quad c_{11} = c_{18},$$

$$b_2 + b_{24} = 0, \quad b_3 + b_{23} = 0, \quad b_6 + b_{22} = 0, \quad b_7 + b_{21} = 0, \quad b_9 + b_{20} = 0, \quad b_{10} + b_{19} = 0, \quad b_{11} + b_{18} = 0.$$

The scheme satisfies the following column simplifying conditions:

$$\sum_{i=j+1}^{25} b_i\,a_{i,j} = b_j\,(1 - c_j), \quad j = 14\,..\,24, \qquad \sum_{i=j+1}^{25} b_i\,c_i\,a_{i,j} = \frac{1}{2}\,b_j\,(1 - c_j^{\,2}), \quad j = 14\,..\,23,$$

$$\sum_{i=j+1}^{25} b_i\,c_i^{\,2}\,a_{i,j} = \frac{1}{3}\,b_j\,(1 - c_j^{\,3}), \quad j = 14\,..\,22, \qquad \sum_{i=j+1}^{25} b_i\,c_i^{\,3}\,a_{i,j} = \frac{1}{4}\,b_j\,(1 - c_j^{\,4}), \quad j = 14\,..\,20,$$

$$\sum_{i=j+1}^{25} b_i\,c_i^{\,4}\,a_{i,j} = \frac{1}{5}\,b_j\,(1 - c_j^{\,5}), \quad j = 14\,..\,17.$$

The zero linking coefficients in the columns 8 to 24 are indicated by the following tableau.

$$\begin{bmatrix}
a_{9,8} \\
a_{10,8} & a_{10,9} \\
a_{11,8} & a_{11,9} & a_{11,10} \\
a_{12,8} & a_{12,9} & a_{12,10} & a_{12,11} \\
0 & a_{13,9} & a_{13,10} & a_{13,11} & a_{13,12} \\
0 & a_{14,9} & a_{14,10} & a_{14,11} & a_{14,12} & a_{14,13} \\
0 & a_{15,9} & a_{15,10} & a_{15,11} & a_{15,12} & a_{15,13} & a_{15,14} \\
0 & a_{16,9} & a_{16,10} & a_{16,11} & a_{16,12} & a_{16,13} & a_{16,14} & a_{16,15} \\
0 & a_{17,9} & a_{17,10} & a_{17,11} & a_{17,12} & a_{17,13} & a_{17,14} & a_{17,15} & a_{17,16} \\
a_{18,8} & a_{18,9} & a_{18,10} & a_{18,11} & a_{18,12} & a_{18,13} & a_{18,14} & a_{18,15} & a_{18,16} & a_{18,17} \\
a_{19,8} & a_{19,9} & a_{19,10} & a_{19,11} & a_{19,12} & a_{19,13} & a_{19,14} & a_{19,15} & a_{19,16} & a_{19,17} & a_{19,18} \\
a_{20,8} & a_{20,9} & a_{20,10} & a_{20,11} & a_{20,12} & a_{20,13} & a_{20,14} & a_{20,15} & a_{20,16} & a_{20,17} & a_{20,18} & a_{20,19} \\
0 & a_{21,9} & a_{21,10} & a_{21,11} & 0 & 0 & 0 & 0 & 0 & 0 & a_{21,18} & a_{21,19} & a_{21,20} \\
0 & a_{22,9} & a_{22,10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{22,19} & a_{22,20} & a_{22,21} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{23,21} & a_{23,22} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{24,23} \\
0 & a_{25,9} & a_{25,10} & a_{25,11} & a_{25,12} & a_{25,13} & a_{25,14} & a_{25,15} & a_{25,16} & a_{25,17} & a_{25,18} & a_{25,19} & a_{25,20} & a_{25,21} & a_{25,22} & a_{25,23} & a_{25,24}
\end{bmatrix}$$

Hiroshi Ono's original scheme also has $a_{19,14} = 0$, $a_{19,15} = 0$ and $a_{20,15} = 0$.
Taking account of the zero linking coefficients, the column simplifying conditions give rise to the following relations between the nodes.

$$c_6 = \frac{3\,c_3 - 1}{2} \quad \text{------------- (v).}$$

$$c_7 = \frac{3\,c_9^{\,2} - 4\,c_6\,c_9 + 2\,c_9 - 2\,c_6 + 1}{4\,c_9 + 2 - 6\,c_6} \quad \text{------------- (vi)}$$

$$c_{11} = \frac{12\,c_{17}^{\,3} - 15\,c_{10}\,c_{17}^{\,2} + 9\,c_{17}^{\,2} - 15\,c_8\,c_{17}^{\,2} - 10\,c_{17}\,c_{10} + 6\,c_{17} + 20\,c_{10}\,c_8\,c_{17} - 10\,c_8\,c_{17} + 3 - 5\,c_8 + 10\,c_{10}\,c_8 - 5\,c_{10}}{5\,(3\,c_{17}^{\,2} + 2\,c_{17} - 4\,c_{17}\,c_{10} - 4\,c_8\,c_{17} - 2\,c_{10} + 1 - 2\,c_8 + 6\,c_{10}\,c_8)} \quad \text{------------- (vii).}$$

These equations make use of the symmetry relations $c_{18} = c_{11}$, $c_{19} = c_{10}$, $c_{20} = c_9$, $c_{21} = c_7$, $c_{22} = c_6$, $c_{23} = c_3$ and the fact that $c_{25} = 1$.
The relations (i), (ii) and (v) give rise to other relations as follows.
Substituting for c_3 from (i) in (v) gives

$$c_6 = c_4 - \frac{1}{2} \quad \text{----- (iv).}$$

Substituting for c_6 from (iv) in (ii) gives

$$c_5 = \frac{(2\,c_4 + 3)\,(2\,c_4 - 1)}{8\,(c_4 + 1)} \quad \text{------- (viii).}$$

We now specify the nodes

$$c_2 = \frac{47}{108},\ c_4 = \frac{101}{152},\ c_9 = \frac{1}{5},\ c_{10} = \frac{1}{3}.$$

The relations (i) to (viii) enable all the nodes to be obtained. Thus we have

$$c_2 = \frac{47}{108},\quad c_3 = \frac{101}{228},\quad c_4 = \frac{101}{152},\quad c_5 = \frac{8225}{76912},\quad c_6 = \frac{25}{152},\quad c_7 = \frac{2013}{3445},\quad c_8 = \frac{17727}{277750},\quad c_9 = \frac{1}{5},\quad c_{10} = \frac{1}{3},$$

$$c_{11}=\frac{51789075}{64972747}+\frac{333240\sqrt{15}}{64972747}+\frac{3057458\sqrt{495+66\sqrt{15}}}{714700217}+\frac{70408\sqrt{495+66\sqrt{15}}\sqrt{15}}{714700217},$$

$$c_{12}=\frac{2523614583531}{6408473250305}+\frac{33872310083\sqrt{15}}{1281694650061}-\frac{1080276228947\sqrt{495+66\sqrt{15}}}{70493205753355}-\frac{48827895154\sqrt{495+66\sqrt{15}}\sqrt{15}}{211479617260065},$$

$$c_{13}=\frac{1}{2}-\frac{\sqrt{495+66\sqrt{15}}}{66},\ c_{14}=\frac{1}{2}-\frac{\sqrt{495-66\sqrt{15}}}{66},\ c_{15}=\frac{1}{2},\ c_{16}=\frac{1}{2}+\frac{\sqrt{495-66\sqrt{15}}}{66},\ c_{17}=\frac{1}{2}+\frac{\sqrt{495+66\sqrt{15}}}{66},$$

with the remaining nodes determined by the symmetry relations $c_2=c_{24},\ c_3=c_{23},\ c_6=c_{22},\ c_7=c_{21},\ c_9=c_{20},\ c_{10}=c_{19},\ c_{11}=c_{18}$.

Step 3:

We specify that $a_{12,10}=-\frac{17}{274},\ a_{12,11}=\frac{2}{6555}$. Then using

$$a_{i,2}=0,\ i=4..13,\quad a_{i,3}=0,\ i=6..13,\quad a_{i,4}=0,\ i=8..13,\quad a_{i,5}=0,\ i=9..13,\quad a_{11,6}=0,\quad a_{13,j}=0,\ j=6..8,$$

the stage-order conditions determine all the linking coefficients in stages 2 to 13.
If exact arithmetic is used, these coefficients are expressed in terms of the radical expressions $\sqrt{15}$ and $\sqrt{495+66\sqrt{15}}$. For example,

$$a_{13,9}=\frac{22650501572361556636801125}{142012394587255301378037l2}-\frac{16238425232388301519 8625\sqrt{15}}{106509295940441476033527 84}-\frac{12096326711671952496 9875\sqrt{495+66\sqrt{15}}}{2603560567433013858597 3472}+\frac{12238378998215586811 5425\sqrt{495+66\sqrt{15}}\sqrt{15}}{4686409021379424945475 22496}.$$

Step 4:

We specify the weights

$$b_2=-\frac{11}{100},\ b_3=-\frac{17}{100},\ b_6=-\frac{19}{100},\ b_7=-\frac{21}{100},\ b_9=-\frac{23}{100},\ b_{10}=-\frac{27}{100},\ b_{11}=-\frac{29}{100},$$

and the linking coefficients

$$a_{19,14}=\frac{191}{846},\quad a_{20,14}=\frac{42}{331},\quad a_{20,15}=\frac{10}{489}.$$

In addition, we specify that

$$a_{i,2}=0,\ i=14..20,\quad a_{i,3}=0,\ i=14..20,\quad a_{i,4}=0,\ i=14..20,\quad a_{i,5}=0,\ i=14..20,\quad a_{i,6}=0,\ i=14..18,$$
$$a_{i,7}=0,\ i=14..17,\quad a_{i,8}=0,\ i=14..17,\quad a_{21,j}=0,\ j=2,3,7,8,12..17,\quad a_{22,j}=0,\ j=2,3,6,8,11..18,$$
$$a_{23,j}=0,\ j=3,4,5,8..20,\quad a_{24,j}=0,\ j=2,4..22,\quad a_{25,j}=0,\ j=4,5,8,$$

We make use of the following symmetry conditions

$$c_2=c_{24},\quad c_3=c_{23},\quad c_6=c_{22},\quad c_7=c_{21},\quad c_9=c_{20},\quad c_{10}=c_{19},\quad c_{11}=c_{18},$$
$$b_2+b_{24}=0,\quad b_3+b_{23}=0,\quad b_6+b_{22}=0,\quad b_7+b_{21}=0,\quad b_9+b_{20}=0,\quad b_{10}+b_{19}=0,\quad b_{11}+b_{18}=0,$$

to determine additional nodes and weights.

The following column simplifying conditions can be used to calculate linking coefficients in columns 14 to 25, so that all the linking coefficients in columns 14 to 24 have their values determined.

$$\sum_{i=j+1}^{25} b_i\,a_{i,j}=b_j\,(1-c_j),\quad j=14..24\qquad \sum_{i=j+1}^{25} b_i\,c_i\,a_{i,j}=\frac{1}{2}\,b_j\,(1-c_j^2),\quad j=14..23,$$

$$\sum_{i=j+1}^{25} b_i\,c_i^2\,a_{i,j}=\frac{1}{3}\,b_j\,(1-c_j^3),\quad j=14..21,\qquad \sum_{i=j+1}^{25} b_i\,c_i^3\,a_{i,j}=\frac{1}{4}\,b_j\,(1-c_j^4),\quad j=14..19,$$

$$\sum_{i=j+1}^{25} b_i\, c_i^4\, a_{i,j} = \frac{1}{5}\, b_j\, (1 - c_j^5), \quad j = 14\,..\,16.$$

Step 5:
The remaining linking coefficients can be determined by using the symmetry realations

$$a_{18,7} = a_{11,7}, \quad a_{18,8} = a_{11,8}, \quad a_{19,6} = a_{10,6}, \quad a_{19,7} = a_{10,7}, \quad a_{19,8} = a_{10,8}, \quad a_{20,6} = a_{9,6}, \quad a_{20,7} = a_{9,7},$$
$$a_{20,8} = a_{9,8}, \quad a_{21,4} = a_{7,4}, \quad a_{21,5} = a_{7,5}, \quad a_{21,6} = a_{7,6}, \quad a_{23,2} = a_{3,2}, \quad a_{22,4} = a_{6,4}, \quad a_{22,5} = a_{6,5},$$

$$a_{25,2} + a_{25,24} = 0, \quad a_{25,3} + a_{25,23} = 0, \quad a_{25,6} + a_{25,22} = 0, \quad a_{25,7} + a_{25,21} = 0,$$
$$a_{24,3} + a_{24,23} = 0, \quad a_{23,7} + a_{23,21} = 0, \quad a_{23,6} + a_{23,22} = 0, \quad a_{22,7} + a_{22,21} = 0, \quad a_{22,9} + a_{22,20} = 0,$$
$$a_{22,10} + a_{22,19} = 0, \quad a_{21,9} + a_{21,20} = 0, \quad a_{21,10} + a_{21,19} = 0, \quad a_{21,11} + a_{21,18} = 0.$$

the stage-order conditions for the stages 14 to 25, namely,

$$\sum_{j=2}^{i-1} a_{i,j} = c_i, \quad i = 14\,..\,25, \qquad \sum_{j=2}^{i-1} a_{i,j}\, c_j = \frac{1}{2}\, c_i^2, \quad i = 14\,..\,20, 25, \qquad \sum_{j=2}^{i-1} a_{i,j}\, c_j^2 = \frac{1}{3}\, c_i^3, \quad i = 14\,..\,20, 25,$$

$$\sum_{j=2}^{i-1} a_{i,j}\, c_j^3 = \frac{1}{4}\, c_i^4, \quad i = 14\,..\,20, 25, \qquad \sum_{j=2}^{i-1} a_{i,j}\, c_j^4 = \frac{1}{5}\, c_i^5, \quad i = 14\,..\,20, 25, \qquad \sum_{j=2}^{i-1} a_{i,j}\, c_j^5 = \frac{1}{6}\, c_i^6, \quad i = 14\,..\,17, 25,$$

and the three extra conditions

$$\sum_{j=2}^{17} a_{18,j}\, c_j^5 = \sum_{j=2}^{10} a_{11,j}\, c_j^5, \quad \sum_{j=2}^{18} a_{19,j}\, c_j^5 = \sum_{j=2}^{9} a_{10,j}\, c_j^5, \quad \sum_{j=2}^{19} a_{20,j}\, c_j^5 = \sum_{j=2}^{8} a_{9,j}\, c_j^5.$$

Once an order 12 scheme has been constructed, an embedded scheme order 9 scheme can be constructed as follows.

We remove rows 13 to 25 from the order 12 scheme and add four new rows.
We specify the following nodes

$$c_{14} = \frac{37}{46}, \; c_{15} = \frac{34}{39}, \; c_{16} = 1,$$

and zero weights

$$b^*_i = 0, \quad i = 2\,..\,8.$$

We also specify the following zero linking coefficients

$$a_{13,j} = 0, \quad j = 2\,..\,5, \quad a_{14,j} = 0, \quad j = 2\,..\,5, \quad a_{15,j} = 0, \quad j = 2\,..\,5, \quad a_{16,j} = 0, \quad j = 2\,..\,5.$$

Step 6:
The nodes c_9, c_{10}, c_{11}, c_{12}, c_{13} are related by the equation

$$\int_0^1 \mathrm{p}(x) \left(\frac{(1-x)^3}{3!}\right) dx \int_0^1 \mathrm{q}(x)\,(1-x)\, dx = \int_0^1 \mathrm{p}(x) \left(\frac{(1-x)^2}{2!}\right) dx \int_0^1 \mathrm{q}(x) \left(\frac{(1-x)^2}{2!}\right) dx,$$

where $\mathrm{p}(x) = x\,(x - c_9)\,(x - c_{10})\,(x - c_{11})\,(x - c_{12})$ and $\mathrm{q}(x) = (x - c_{13})\,\mathrm{p}(x)$.
See: J.H. Verner, SIAM Journal of Numerical Analysis 1978, 772-790, "Explicit Runge-Kutta methods with estimates of the Local Truncation Error." (page 780)
This relation gives

$$c_{13} \simeq 0.4970267001007476028032930885363848318550815967070143979104342007017543082576391973337$$

Step 7:
The quadrature conditions

$$\sum_{i=1}^{16} b^*_i = 1, \quad \sum_{i=1}^{16} b^*_i\, c_i^k = \frac{1}{k+1}, \quad k = 1\,..\,8,$$

can be used to determine the weights b^*_1, b^*_9, b^*_{10}, b^*_{11}, b^*_{12}, b^*_{13}, b^*_{14}, b^*_{15} and b^*_{16}.

Step 8:
We require that stages 13 to 16 have stage-order 5 so that

$$\sum_{j=1}^{i-1} a_{i,j} = c_i, \qquad \sum_{j=2}^{i-1} a_{i,j}\, c_j^{(k-1)} = \frac{1}{k}\, c_i^{\,k}, \quad k = 2\,..\,5,\; i = 13\,..\,16.$$

These equations can be used to obtain expressions for the 20 linking coefficients

$$a_{13,1},\; a_{13,7},\; a_{13,8},\; a_{13,9},\; a_{13,12},\; a_{14,1},\; a_{14,8},\; a_{14,9},\; a_{14,12},\; a_{14,13},$$
$$a_{15,6},\; a_{15,7},\; a_{15,10},\; a_{15,11},\; a_{15,14},\; a_{16,1},\; a_{16,8},\; a_{16,9},\; a_{16,12},\; a_{16,13}$$

in terms of the other linking coefficients.

Step 9:
The column simplifying conditions

$$\sum_{i=2}^{16} b^*_i\, a_{i,1} = b^*_1, \qquad \sum_{i=j+1}^{16} b^*_i\, a_{i,j} = b^*_j\,(1 - c_j), \quad j = 8,\, 12\,..\,15.$$

can be used to obtain expressions for the 6 linking coefficients

$$a_{15,8},\; a_{15,9},\; a_{15,13},\; a_{16,7},\; a_{16,14},\; a_{16,15},$$

in terms of the linking coefficients

$$a_{13,6},\; a_{13,10},\; a_{13,11},\; a_{14,6},\; a_{14,7},\; a_{14,10},\; a_{14,11},\; a_{15,1},\; a_{15,12},\; a_{16,6},\; a_{16,10},\; a_{16,11}.$$

Step 10:
A system of four equations arising from the order conditions

$$\sum_{i=5}^{16} b^*_i\, c_i^{\,2} \left(\sum_{j=4}^{i-1} a_{i,j} \left(\sum_{k=3}^{j-1} a_{j,k} \left(\sum_{l=2}^{k-1} a_{k,l}\, c_l^{\,3} \right)\right)\right) = \frac{1}{1080}, \qquad \sum_{i=4}^{16} b^*_i\, c_i^{\,2} \left(\sum_{j=3}^{i-1} a_{i,j} \left(\sum_{k=2}^{j-1} a_{j,k}\, c_k^{\,4} \right)\right) = \frac{1}{270},$$

$$\sum_{i=3}^{16} b^*_i\, c_i^{\,2} \left(\sum_{j=2}^{i-1} a_{i,j}\, c_j^{\,5} \right) = \frac{1}{54}, \qquad \sum_{i=3}^{16} b^*_i\, c_i \left(\sum_{j=2}^{i-1} a_{i,j}\, c_j^{\,6} \right) = \frac{1}{63}$$

can be solved to give linear expressions for the 4 linking coefficients $a_{15,12},\; a_{16,6},\; a_{16,11},\; a_{16,10}$ in terms of other linking coefficients.

Step 11:
A system of three equations arising from the order conditions

$$\sum_{i=5}^{16} b^*_i \left(\sum_{j=4}^{i-1} a_{i,j} \left(\sum_{k=3}^{j-1} a_{j,k} \left(\sum_{l=2}^{k-1} a_{k,l}\, c_l^{\,4} \right)\right)\right) = \frac{1}{1680}, \qquad \sum_{i=3}^{16} b^*_i\, c_i \left(\sum_{j=2}^{i-1} a_{i,j}\, c_j^{\,5} \right) = \frac{1}{48},$$

$$\sum_{i=6}^{16} b^*_i\, c_i^{\,2} \left(\sum_{j=5}^{i-1} a_{i,j} \left(\sum_{k=4}^{j-1} a_{j,k} \left(\sum_{l=3}^{k-1} a_{k,l} \left(\sum_{m=2}^{l-1} a_{l,m}\, c_m^{\,2} \right)\right)\right)\right) = \frac{1}{3240}$$

can be solved to give linear expressions for the 3 linking coefficients $a_{14,7},\; a_{14,10},\; a_{15,1}$ in terms of other linking coefficients.

Step 12:
An equation arising from the order condition

$$\sum_{i=4}^{16} b^*_i\, c_i \left(\sum_{j=3}^{i-1} a_{i,j}\, c_j \left(\sum_{k=2}^{j-1} a_{j,k}\, c_k^{\,4} \right)\right) = \frac{1}{315}$$

can be solved to obtain expression for the linking coefficient $a_{14,11}$ in terms of other linking coefficients.

Step 13:
An equation arising from the order condition

$$\sum_{i=5}^{16} b*_i\, c_i \left(\sum_{j=4}^{i-1} a_{i,j}\, c_j \left(\sum_{k=3}^{j-1} a_{j,k} \left(\sum_{l=2}^{k-1} a_{k,l}\, c_l^{\,3}\right)\right)\right) = \frac{1}{1260}$$

can be solved to obtain expression for the linking coefficient $a_{14,6}$ in terms of $a_{13,6}$.

Step 14:
At this stage the only remaining unknown linking coefficients are $a_{13,6}$, $a_{13,10}$ and $a_{13,11}$.
A system of three equations arising from the order conditions

$$\sum_{i=6}^{16} b*_i\, c_i \left(\sum_{j=5}^{i-1} a_{i,j} \left(\sum_{k=4}^{j-1} a_{j,k} \left(\sum_{l=3}^{k-1} a_{k,l} \left(\sum_{m=2}^{l-1} a_{l,m}\, c_m^{\,3}\right)\right)\right)\right) = \frac{1}{7560},$$

$$\sum_{i=5}^{16} b*_i\, c_i \left(\sum_{j=4}^{i-1} a_{i,j} \left(\sum_{k=3}^{j-1} a_{j,k} \left(\sum_{l=2}^{k-1} a_{k,l}\, c_l^{\,4}\right)\right)\right) = \frac{1}{1890}, \qquad \sum_{i=4}^{16} b*_i\, c_i \left(\sum_{j=3}^{i-1} a_{i,j} \left(\sum_{k=2}^{j-1} a_{j,k}\, c_k^{\,5}\right)\right) = \frac{1}{378}$$

can be used to find values for these three linking coefficients.

The order 9 scheme can be embedded in the order 12 scheme by inserting the 13th, 14th, 15th and 16th rows as 26th, 27th, 28th and 29th rows of a combined scheme respectively (with the some horizontal adjustments and addition of zero coefficients).
In detail, the nodes c_{13}, c_{14}, c_{15} and c_{16} of the 16 stage, order 9 scheme become the nodes c_{26}, c_{27}, c_{28} and c_{29} of the combined scheme respectively, and the first 12 entries in the 13th, 14th, 15th and 16th rows of the order 9 scheme are inserted as the first 12 entries of rows 26, 27, 28 and 29 of the combined scheme respectively. The linking coefficients $a_{26,13}$ to $a_{26,25}$ are zero as are $a_{27,13}$ to $a_{27,25}$, $a_{28,13}$ to $a_{28,25}$ and $a_{29,13}$ to $a_{29,25}$. The linking coefficients $a_{27,26}$ of the combined scheme is $a_{14,13}$ of the 16 stage order 9 scheme. The linking coefficients $a_{28,26}$ and $a_{28,27}$ of the combined scheme are the linking coefficients $a_{15,13}$ and $a_{15,14}$ respectively of the 16 stage order 9 scheme. The linking coefficients $a_{29,26}$, $a_{29,27}$ and $a_{29,28}$ of the combined scheme are the linking coefficients $a_{16,13}$, $a_{16,14}$ and $a_{16,15}$ respectively of the 16 stage order 9 scheme.

Similarly, the weights $b*_1$ to $b*_{12}$ of the order 9 scheme become the first 12 weights of the embedded scheme and the weights $b*_{13}$ to $b*_{25}$ are zero. The weights $b*_{26}$, $b*_{27}$, $b*_{28}$ and $b*_{29}$ of the combined scheme are the weights $b*_{13}$, $b*_{14}$, $b*_{15}$ and $b*_{16}$ of the 16 stage order 9 scheme respectively.

We can set $b_{26}=0$, $b_{27}=0$, $b_{28}=0$ and $b_{29}=0$ to make the order 12 scheme into a 29 stage scheme.

The principal error norm of an order 12 scheme constructed in the manner described can be calculated using 50 of the 12486 principal error terms. These error terms are given in an abreviated form as follows.

$$b\,c\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,c))))))))))-\frac{1}{518918400}, \quad b\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,c)))))))))\,(a\,c)-\frac{1}{94348800},$$

$$b\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,c))))))))\,(a\,(a\,c))-\frac{1}{28304640}, \quad \frac{1}{2}\left(b\,c\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,c^2)))))))))-\frac{1}{259459200}\right),$$

$$\frac{1}{2}\left(b\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,c^2))))))))\,(a\,c)-\frac{1}{47174400}\right), \quad \frac{1}{2}\left(b\,c^2\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,c)))))))))-\frac{1}{47174400}\right),$$

$$\frac{1}{2}\left(b\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,c^2)))))))\,(a\,(a\,c))-\frac{1}{14152320}\right), \quad \frac{1}{2}\left(b\,(a\,c^2)\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,c))))))))-\frac{1}{14152320}\right),$$

$$\frac{1}{4}\left(b\,c^2\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,c^2)))))))) - \frac{1}{23587200}\right),$$

$$\frac{1}{4}\left(b\,(c^2\,a)\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,c^2))))))) - \frac{1}{7076160}\right),$$

$$\frac{1}{6}\left(b\,c\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,c^3)))))))) - \frac{1}{86486400}\right),$$

$$\frac{1}{6}\left(b\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,c^3)))))))\,(a\,c) - \frac{1}{15724800}\right),$$

$$\frac{1}{6}\left(b\,(a\,(a\,(a\,(a\,(a\,(a\,c^3))))))\,(a\,(a\,c)) - \frac{1}{4717440}\right)),$$

$$\frac{1}{6}\left(b\,c^3\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,c)))))))) - \frac{1}{4717440}\right),$$

$$\frac{1}{12}\left(b\,c^2\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,c^3))))))) - \frac{1}{7862400}\right),$$

$$\frac{1}{12}\left(b\,(a\,(a\,(a\,(a\,(a\,(a\,c^3))))))\,(a\,c^2) - \frac{1}{2358720}\right),$$

$$\frac{1}{12}\left(b\,c^3\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,c^2))))))) - \frac{1}{2358720}\right),$$

$$\frac{1}{24}\left(b\,c\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,c^4))))))) - \frac{1}{21621600}\right),$$

$$\frac{1}{24}\left(b\,(a\,(a\,(a\,(a\,(a\,(a\,c^4))))))\,(a\,c) - \frac{1}{3931200}\right),$$

$$\frac{1}{24}\left(b\,(a\,(a\,(a\,(a\,(a\,c^4)))))\,(a\,(a\,c)) - \frac{1}{1179360}\right),$$

$$\frac{1}{24}\left(b\,c^4\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,c))))))) - \frac{1}{524160}\right),$$

$$\frac{1}{36}\left(b\,c^3\,(a\,(a\,(a\,(a\,(a\,(a\,c^3)))))) - \frac{1}{786240}\right),$$

$$\frac{1}{48}\left(b\,c^2\,(a\,(a\,(a\,(a\,(a\,(a\,c^4)))))) - \frac{1}{1965600}\right),$$

$$\frac{1}{48}\left(b\,(a\,(a\,(a\,(a\,(a\,c^4)))))\,(a\,c^2) - \frac{1}{589680}\right),$$

$$\frac{1}{48}\left(b\,c^4\,(a\,(a\,(a\,(a\,(a\,(c^2\,a)))))) - \frac{1}{262080}\right),$$

$$\frac{1}{120}\left(b\,c\,(a\,(a\,(a\,(a\,(a\,(a\,c^5)))))) - \frac{1}{4324320}\right),$$

$$\frac{1}{120}\left(b\,(a\,c)\,(a\,(a\,(a\,(a\,(a\,c^5))))) - \frac{1}{786240}\right),$$

$$\frac{1}{120}\left(b\,(a\,(a\,c))\,(a\,(a\,(a\,(a\,c^5)))) - \frac{1}{235872}\right),$$

$$\frac{1}{120}\left(b\,c^5\,(a\,(a\,(a\,(a\,(a\,(a\,c)))))) - \frac{1}{65520}\right),$$

$$\frac{1}{144}\left(b\,c^3\,(a\,(a\,(a\,(a\,(a\,c^4))))) - \frac{1}{196560}\right),$$

$$\frac{1}{144}\left(b\,c^4\,(a\,(a\,(a\,(a\,(a\,c^3))))) - \frac{1}{87360}\right),$$

$$\frac{1}{240}\left(b\,c^2\,(a\,(a\,(a\,(a\,(a\,c^5))))) - \frac{1}{393120}\right),$$

$$\frac{1}{240}\left(b\,(a\,c^2)\,(a\,(a\,(a\,(a\,c^5)))) - \frac{1}{117936}\right),$$

$$\frac{1}{240}\left(b\,c^5\,(a\,(a\,(a\,(a\,(a\,c^2))))) - \frac{1}{32760}\right),$$

$$\frac{1}{720}\left(b\,c\,(a\,(a\,(a\,(a\,(a\,c^6))))) - \frac{1}{720720}\right),$$

$$\frac{1}{720}\left(b\,(a\,(a\,(a\,(a\,c^6))))\,(a\,c) - \frac{1}{131040}\right),$$

$$\frac{1}{720}\left(b\,(a\,(a\,(a\,c^6)))\,(a\,(a\,c)) - \frac{1}{39312}\right),$$

$$\frac{1}{576}\left(b\,c^4\,(a\,(a\,(a\,(a\,c^4)))) - \frac{1}{21840}\right),$$

$$\frac{1}{720}\left(b\,c^3\,(a\,(a\,(a\,(a\,c^5)))) - \frac{1}{39312}\right),$$

$$\frac{1}{720}\left(b\,c^5\,(a\,(a\,(a\,(a\,c^3)))) - \frac{1}{10920}\right),$$

$$\frac{1}{1440}\left(b\,c^2\,(a\,(a\,(a\,(a\,c^6)))) - \frac{1}{65520}\right),$$

$$\frac{1}{1440}\left(b\,(a\,c^2)\,(a\,(a\,(a\,c^6))) - \frac{1}{19656}\right),$$

$$\frac{1}{2880}\left(b\,c^4\,(a\,(a\,(a\,c^5))) - \frac{1}{4368}\right),$$

$$\frac{1}{2880}\left(b\,c^5\,(a\,(a\,(a\,c^4))) - \frac{1}{2730}\right),$$

$$\frac{1}{4320}\left(b\,c^3\ (a\ (a\ (a\,c^6))) - \frac{1}{6552}\right), \qquad \frac{1}{14400}\left(b\,c^5\ (a\ (a\,c^5)) - \frac{1}{546}\right), \qquad \frac{1}{17280}\left(b\,c^4\ (a\ (a\,c^6)) - \frac{1}{728}\right),$$

$$\frac{1}{86400}\left(b\,c^5\ (a\,c^6) - \frac{1}{91}\right), \qquad \frac{1}{39916800}\left(b\ (a\,c^{11}) - \frac{1}{156}\right), \qquad \frac{1}{479001600}\left(b\,c^{12} - \frac{1}{13}\right).$$

For example, $\frac{1}{1440}\left(b\ (a\,c^2)\ (a\ (a\ (a\,c^6))) - \frac{1}{19656}\right)$ is an abreviation for

$$\frac{1}{1440}\left(\left(\sum_{i=5}^{25} b_i \left(\sum_{j=2}^{i-1} a_{i,j}\,c_j^{\,2}\right)\left(\sum_{j=4}^{i-1} a_{i,j}\left(\sum_{k=3}^{j-1} a_{j,k}\left(\sum_{l=2}^{k-1} a_{k,l}\,c_l^{\,6}\right)\right)\right)\right) - \frac{1}{19656}\right).$$

The principal error norm can be calculated as $\sqrt{\sum_{i=1}^{50} d_i\,e_i^{\,2}}$, where e_i is the value of the ith error term above and d_i is the ith member of the sequence

2, 1, 1, 2, 1, 5, 1, 1, 5, 1, 20, 10, 10, 46, 50, 10, 46, 156, 78, 78, 554, 460, 390, 78, 554, 2052, 1026, 1026, 8766, 3588, 5540, 5130, 1026, 8766, 39432, 19716, 19716, 43212, 47196, 87660, 98580, 19716, 568404, 683748, 906936, 8993916, 10922664, 172830456, $\frac{908488322227355}{144}$, 1.

The principal error norm of an order 9 embedded scheme constructed in the manner described can be calculated using 24 of the 719 principal error terms. These error terms are given in an abreviated form as follows.

$$b^*\,c\ (a\ (a\ (a\ (a\ (a\ (a\ (a\,c))))))) - \frac{1}{403200}, \qquad b^*\,c\ (a\,c\ (a\ (a\ (a\ (a\ (a\,c)))))) - \frac{1}{57600},$$

$$\frac{1}{2}\left(b^*\,c\ (a\ (a\ (a\ (a\ (a\ (a\,c^2)))))) - \frac{1}{201600}\right), \qquad \frac{1}{2}\left(b^*\,c^2\ (a\ (a\ (a\ (a\ (a\ (a\,c)))))) - \frac{1}{50400}\right),$$

$$\frac{1}{2}\left(b^*\,c\ (a\,c\ (a\ (a\ (a\ (a\,c^2))))) - \frac{1}{28800}\right), \qquad \frac{1}{4}\left(b^*\,c^2\ (a\ (a\ (a\ (a\ (a\,c^2))))) - \frac{1}{25200}\right),$$

$$\frac{1}{6}\left(b^*\,c\ (a\ (a\ (a\ (a\ (a\,c^3))))) - \frac{1}{67200}\right), \qquad \frac{1}{6}\left(b^*\,c^3\ (a\ (a\ (a\ (a\ (a\,c))))) - \frac{1}{7200}\right),$$

$$\frac{1}{6}\left(b^*\,c\ (a\,c\ (a\ (a\ (a\,c^3)))) - \frac{1}{9600}\right), \qquad \frac{1}{12}\left(b^*\,c^2\ (a\ (a\ (a\ (a\,c^3)))) - \frac{1}{8400}\right),$$

$$\frac{1}{12}\left(b^*\,c^3\ (a\ (a\ (a\ (a\,c^2)))) - \frac{1}{3600}\right), \qquad \frac{1}{24}\left(b^*\,c\ (a\ (a\ (a\ (a\,c^4)))) - \frac{1}{16800}\right),$$

$$\frac{1}{36}\left(b^*\,c^3\ (a\ (a\ (a\,c^3))) - \frac{1}{1200}\right), \qquad \frac{1}{24}\left(b^*\,c\ (a\,c\ (a\ (a\,c^4))) - \frac{1}{2400}\right),$$

$$\frac{1}{48}\left(b^*\,c^2\ (a\ (a\ (a\,c^4))) - \frac{1}{2100}\right), \qquad \frac{1}{120}\left(b^*\,c\ (a\ (a\ (a\,c^5))) - \frac{1}{3360}\right),$$

$$\frac{1}{144}\left(b^*\,c^3\ (a\ (a\,c^4)) - \frac{1}{300}\right), \qquad \frac{1}{120}\left(b^*\,c\ (a\,c\ (a\,c^5)) - \frac{1}{480}\right), \qquad \frac{1}{240}\left(b^*\,c^2\ (a\ (a\,c^5)) - \frac{1}{420}\right),$$

$$\frac{1}{720}\left(b^*\,c\ (a\ (a\,c^6)) - \frac{1}{560}\right), \qquad \frac{1}{720}\left(b^*\,c^3\ (a\,c^5) - \frac{1}{60}\right), \qquad \frac{1}{1440}\left(b^*\,c^2\ (a\,c^6) - \frac{1}{70}\right),$$

$$\frac{1}{5040}\left(b^*\,c\ (a\,c^7) - \frac{1}{80}\right), \qquad \frac{1}{362880}\left(b^*\,c^9 - \frac{1}{10}\right).$$

The principal error norm can be calculated as $\sqrt{\sum_{i=1}^{24} \delta_i \, {e^*_i}^2}$, where e^*_i is the value of the ith error term above and δ_i is the ith member of the sequence

2, 2, 2, 6, 2, 6, 20, 30, 20, 60, 30, 156, 300, 156, 468, 2052, 2340, 2052, 6156, 39432, 30780, 118296, 840744, 1295337252.

For example, $\frac{1}{120}\left(b^* c\ (a\ (a\ (a\, c^5))) - \frac{1}{3360}\right)$ is an abreviation for

$$\frac{1}{120}\left(\left(\sum_{i=5}^{16} b^*_i\, c_i\left(\sum_{j=4}^{i-1} a_{i,j}\left(\sum_{k=3}^{j-1} a_{j,k}\left(\sum_{l=2}^{k-1} a_{k,l}\, c_l^5\right)\right)\right)\right) - \frac{1}{3360}\right).$$

The complete 29 stage combined order 9 and 12 Runge-Kutta scheme is determined by the parameter values

$$c_2 = \frac{47}{108},\ c_4 = \frac{101}{152},\ c_9 = \frac{1}{5},\ c_{10} = \frac{1}{3},\ c_{25} = 1,\ a_{12,10} = -\frac{17}{274},\ a_{12,11} = \frac{2}{6555},\ a_{19,14} = \frac{191}{846},\ a_{20,14} = \frac{42}{331},\ a_{20,15} = \frac{10}{489},$$

$$b_2 = -\frac{11}{100},\ b_3 = -\frac{17}{100},\ b_6 = -\frac{19}{100},\ b_7 = -\frac{21}{100},\ b_9 = -\frac{23}{100},\ b_{10} = -\frac{27}{100},\ b_{11} = -\frac{29}{100},\ c_{27} = \frac{37}{46},\ c_{28} = \frac{34}{39},\ c_{29} = 1.$$

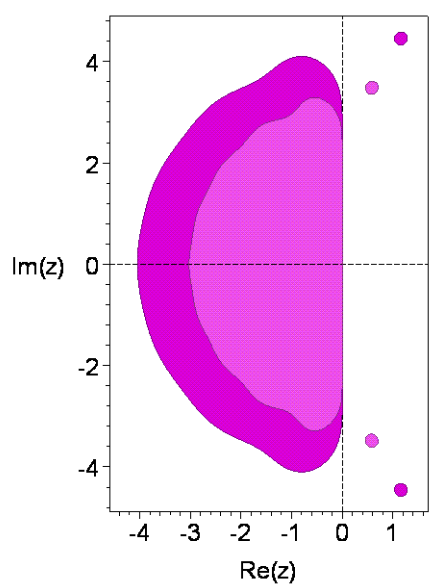
The weights of the current order 12 scheme are the same as those of Hiroshi Ono's scheme, but his scheme has $c_2 = \frac{1}{4}$, $c_4 = \frac{2}{3}$, $a_{19,14} = 0$, $a_{20,14} = 0$, $a_{20,15} = 0$. In addition he specifies that $a_{12,6} = 0$ and $a_{12,7} = 0$, whereas these coefficients are nonzero in the current scheme associated with the values chosen for $a_{12,10}$ and $a_{12,11}$. Hiroshi Ono makes no mention of an embedded scheme.

The principal error norm of the current order 12 scheme is $0.3152572305 \times 10^{(-7)}$ compared to $0.3097360773 \times 10^{(-6)}$ for Hiroshi Ono's scheme and $0.1367113081 \times 10^{(-6)}$ for a scheme of Terry Feagin.

The principal error norm of the embedded order 9 scheme is $0.7348313900 \times 10^{(-5)}$.

The maximum magnitude of the linking coefficients is 212.1164197 and the 2-norm of the linking coefficients is 384.3703602.

The stability regions for the two schemes are shown in the following picture.



The stability region of the order 9 scheme appears in the darker shade.

The real stability intervals of the order 12 and 9 schemes are respectively $[-3.0248, 0]$ and $[-4.0456, 0]$ and the stability region intersects the nonnegative imaginary axis in the interval $[\,0.7481, 2.4158\, i\,]$.

A complete list of the values of the coefficients correct to 85 digits is given in a separate document.