Construction of 26 stage combined order 10 and 11 Runge-Kutta schemes

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A method for the constuction of combined 10 and 11 Runge-Kutta schemes is described here. The embedded order 10 scheme is provided with the aim of it being used for error control in the standard manner. The method of construction of these schemes is motivated by a 10(9) scheme due to Tom Baker at the University of Teeside.

When the construction reaches the stage of requiring equations derived from the order conditions, the order conditions used at each step were selected from those that remained to be satisfied after the preceding step. Details of how this was achieved can be seen by viewing Maple worksheets which are available online.

The parameters that determine the specific schemes described here were obtained from a random search followed by a lengthy process of minimization of the principal error norm.
The coefficents for the 2 schemes listed here are given in separate documents.

We first indicate how to construct a 25 stage order 11 Runge-Kutta scheme.
Step 1:
The stage-orders of stages 3 to 25 are given as follows.
We specify the following stage-orders for stages 3 to 25.

$$\left[\begin{array}{c|ccccccccccccccccccccccc} \textit{stage} & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 \\ \textit{stage-order} & 3 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 5 & 6 & 6 & 6 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \end{array}\right]$$

We start by considering the rows 2 to 16 in the Butcher tableau of the order 11 scheme.
We have the following stage-order conditions involving the nodes c_i and linking coefficients $a_{i,j}$.

$$\sum_{j=1}^{i-1} a_{i,j} = c_i \text{ for } i = 3\,..\,16, \qquad \sum_{j=2}^{i-1} a_{i,j}\,c_j = \frac{c_i^2}{2} \text{ for } i = 3\,..\,16, \qquad \sum_{j=2}^{i-1} a_{i,j}\,c_j^2 = \frac{c_i^3}{3} \text{ for } i = 3\,..\,16, \qquad \sum_{j=2}^{i-1} a_{i,j}\,c_j^3 = \frac{c_i^4}{4} \text{ for } i = 6\,..\,16,$$

$$\sum_{j=2}^{i-1} a_{i,j}\,c_j^4 = \frac{c_i^5}{5} \text{ for } i = 9\,..\,16, \qquad \sum_{j=2}^{i-1} a_{i,j}\,c_j^5 = \frac{c_i^6}{6} \text{ for } i = 12\,..\,16, \qquad \sum_{j=2}^{i-1} a_{i,j}\,c_j^6 = \frac{c_i^7}{7} \text{ for } i = 15, 16.$$

We also have zero linking coefficients as indicated by the following tableau.

$$\left[\begin{array}{cccccccccccccccc}
c_2 & a_{2,1} \\
c_3 & a_{3,1} & a_{3,2} \\
c_4 & a_{4,1} & 0 & a_{4,3} \\
c_5 & a_{5,1} & 0 & a_{5,3} & a_{5,4} \\
c_6 & a_{6,1} & 0 & 0 & a_{6,4} & a_{6,5} \\
c_7 & a_{7,1} & 0 & 0 & 0 & a_{7,5} & a_{7,6} \\
c_8 & a_{8,1} & 0 & 0 & 0 & a_{8,5} & a_{8,6} & a_{8,7} \\
c_9 & a_{9,1} & 0 & 0 & 0 & 0 & a_{9,6} & a_{9,7} & a_{9,8} \\
c_{10} & a_{10,1} & 0 & 0 & 0 & 0 & a_{10,6} & a_{10,7} & a_{10,8} & a_{10,9} \\
c_{11} & a_{11,1} & 0 & 0 & 0 & 0 & 0 & 0 & a_{11,8} & a_{11,9} & a_{11,10} \\
c_{12} & a_{12,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{12,9} & a_{12,10} & a_{12,11} \\
c_{13} & a_{13,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{13,9} & a_{13,10} & a_{13,11} & a_{13,12} \\
c_{14} & a_{14,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{14,9} & a_{14,10} & a_{14,11} & a_{14,12} & a_{14,13} \\
c_{15} & a_{15,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{15,11} & a_{15,12} & a_{15,13} & a_{15,14} \\
c_{16} & a_{16,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{16,11} & a_{16,12} & a_{16,13} & a_{16,14} & a_{16,15}
\end{array}\right]$$

We can use this information to obtain certain relations involving the nodes c_2 to c_{16}.

$$\int_0^{c_3} x\,(x-c_2)\,dx = 0 \quad \text{or} \quad c_2 = \frac{2}{3}\,c_3 \qquad \text{------------- (i)}$$

$$\int_0^{c_4} x\,(x-c_3)\,dx = 0 \quad \text{or} \quad c_3 = \frac{2}{3}\,c_4 \qquad \text{------------- (ii)}$$

$$\int_0^{c_6} x\,(x-c_4)\,(x-c_5)\,dx = 0 \quad \text{or} \quad c_4 = \frac{c_6\,(3\,c_6-4\,c_5)}{2\,(2\,c_6-3\,c_5)} \qquad \text{------------- (iii)}$$

$$\int_0^{c_9} x\,(x-c_6)\,(x-c_7)\,(x-c_8)\,dx = 0 \quad \text{or} \quad c_6 = \frac{(20\,c_8\,c_7-15\,c_9\,c_8+12\,c_9^{\,2}-15\,c_9\,c_7)\,c_9}{5\,(6\,c_8\,c_7-4\,c_9\,c_8+3\,c_9^{\,2}-4\,c_9\,c_7)} \qquad \text{------------- (iv)}$$

$$\int_0^{c_{11}} x\,(x-c_8)\,(x-c_9)\,(x-c_{10})\,dx = 0 \quad \text{or} \quad c_8 = \frac{(15\,c_9\,c_{11}-20\,c_9\,c_{10}-12\,c_{11}^{\,2}+15\,c_{10}\,c_{11})\,c_{11}}{5\,(4\,c_9\,c_{11}-6\,c_9\,c_{10}-3\,c_{11}^{\,2}+4\,c_{10}\,c_{11})} \qquad \text{------------- (v)}$$

$$\int_0^{c_{12}} x\,(x-c_9)\,(x-c_{10})\,(x-c_{11})\,dx = 0 \qquad \text{------------- (vi)}$$

$$\int_0^{c_{12}} x^2\,(x-c_9)\,(x-c_{10})\,(x-c_{11})\,dx = 0 \qquad \text{------------- (vii)}$$

$$\int_0^{c_{13}} x\,(x-c_9)\,(x-c_{10})\,(x-c_{11})\,(x-c_{12})\,dx = 0 \qquad \text{------------- (viii)}$$

$$\int_0^{c_{15}} x\,(x-c_{11})\,(x-c_{12})\,(x-c_{13})\,(x-c_{14})\,dx = 0 \qquad \text{------------- (ix)}$$

$$\int_0^{c_{15}} x^2\,(x-c_{11})\,(x-c_{12})\,(x-c_{13})\,(x-c_{14})\,dx = 0 \qquad \text{------------- (x)}$$

$$\int_0^{c_{16}} x\,(x-c_{11})\,(x-c_{12})\,(x-c_{13})\,(x-c_{14})\,(x-c_{15})\,dx = 0 \qquad \text{------------- (xi)}$$

Introducing the relation $c_{11} = \frac{2}{3}\,c_{13}$ allows the equations (vi), (vii) and (viii) to produce the relations

$$c_9 = \frac{98-\sqrt{10}+\sqrt{3694-212\sqrt{10}}}{156}\,c_{13}, \qquad c_{10} = \frac{98-\sqrt{10}-\sqrt{3694-212\sqrt{10}}}{156}\,c_{13}$$

and

$$c_{12} = \frac{40-2\sqrt{10}}{39}\,c_{13}.$$

The equations (ix) and (x) in conjunction with the relations $c_{11} = \frac{2}{3}\,c_{13}$ and $c_{12} = \frac{40-2\sqrt{10}}{39}\,c_{13}$ give rise to the relations

$$c_{13} = \eta\,c_{15}$$

and

$$c_{14} = \frac{80\,(\sqrt{10}-20)\,\eta^3+10\,(417-15\sqrt{10})\,\eta^2+36\,(2\sqrt{10}-105)\,\eta+1170}{1404+120\,(\sqrt{10}-20)\,\eta^3+40\,(139-5\sqrt{10})\,\eta^2+45\,(2\sqrt{10}-105)\,\eta}\,c_{15}.$$

where η is a real zero of the degree 6 polynomial

$$\mathrm{P}(z) = 28000\,z^6-(117600+1680\sqrt{10})\,z^5+(6090\sqrt{10}+203910)\,z^4-(188700+8700\sqrt{10})\,z^3+(99969+6276\sqrt{10})\,z^2-(29250+2340\sqrt{10})\,z+3690+360\sqrt{10}.$$

This polynomial has two real zeros with the approximate values 0.46400913396513 and 0.90501790944446.
Equation (xi) gives rise to a relation of the form

$$c_{15} = \theta\, c_{16},$$

where θ is a real zero of the cubic polynomial

$$\mathrm{S}(z) = d_3\, z^3 + d_2\, z^2 + d_1\, z + d_0,$$

with

$$d_0 = 42\,(4000\,\eta^6 - 16800\,\eta^5 - 240\,\eta^5\sqrt{10} + 870\,\eta^4\sqrt{10} + 29130\,\eta^4 - 1500\,\eta^3\sqrt{10} - 32100\,\eta^3 + 26367\,\eta^2 + 1668\,\eta^2\sqrt{10} - 14625\,\eta - 1170\,\eta\sqrt{10} + 3690 + 360\sqrt{10}),$$

$$d_1 = 35\,(4000\,\eta^6 - 240\,\eta^5\sqrt{10} - 16800\,\eta^5 + 48330\,\eta^4 + 1470\,\eta^4\sqrt{10} - 3300\,\eta^3\sqrt{10} - 72300\,\eta^3 + 3258\,\eta^2\sqrt{10} + 52452\,\eta^2 - 1170\,\eta\sqrt{10} - 14625\,\eta),$$

$$d_2 = 200\,(560\,\eta^6 - 84\,\eta^5\sqrt{10} - 5880\,\eta^5 + 12327\,\eta^4 + 357\,\eta^4\sqrt{10} - 462\,\eta^3\sqrt{10} - 10122\,\eta^3 + 2961\,\eta^2 + 189\,\eta^2\sqrt{10}),$$

$$d_3 = 4200\,(80\,\eta^6 - 210\,\eta^5 - 3\,\eta^5\sqrt{10} + 6\,\eta^4\sqrt{10} + 192\,\eta^4 - 3\,\eta^3\sqrt{10} - 60\,\eta^3).$$

We take

$$\eta \simeq 0.9050179094444592302850092001266376783360386842107738556985099105305710627647138909540$$

and choose the largest of the three resulting values for θ, namely

$$\theta \simeq 1.9004604073182104057815740173261605835945164456680765456740452531029401208773933 99745.$$

Specifying the nodes c_7 and c_{13} allows all the nodes and linking coefficients in stages 2 to 16 to be calculated.

Step 2:
We specify the nodes c_{17}, c_{18}, c_{19}, the linking coefficients $a_{18,17}$, $a_{19,17}$, $a_{19,18}$ and the zero linking coefficients

$$a_{i,j} = 0, \quad j = 2\,..\,10, \; i = 17, 18, 19.$$

The stages 17, 18 and 19 have stage-order 7. Thus the following conditions hold.

$$\sum_{j=1}^{i-1} a_{i,j} = c_i \;\text{ for } i = 17, 18, 19, \qquad \sum_{j=2}^{i-1} a_{i,j}\, c_j^{\,k} = \frac{c_i^{(k+1)}}{k+1} \;\text{ for } i = 17, 18, 19,\; k = 1\,..\,6.$$

This allows all the linking coefficients in stages 17, 18 and 19 to be obtained.

Step 3:
We specify the remaining nodes c_{20}, c_{21}, c_{22}, c_{23}, c_{24}, $c_{25} = 1$ and the weight b_{25}.
We also require that $b_i = 0$, $i = 2\,..\,14$. Then we can obtain all the remaining weights by using the order 11 quadrature conditions:

$$\sum_{i=1}^{25} b_i = 1, \quad \sum_{i=1}^{25} b_i\, c_i^{\,k} = \frac{1}{k+1}, \quad k = 1\,..\,10.$$

Step 4:
The stages 20 to 25 all have stage-order 7. Thus the following conditions hold.

$$\sum_{j=1}^{i-1} a_{i,j} = c_i \;\text{ for } i = 20\,..\,25, \qquad \sum_{j=2}^{i-1} a_{i,j}\, c_j^{\,k} = \frac{c_i^{(k+1)}}{k+1} \;\text{ for } i = 20\,..\,25,\; k = 1\,..\,6.$$

After specifying the zero linking coefficients $a_{i,j} = 0$ for $j = 2\,..\,10$, $i = 20\,..\,25$, these equations provide linear relations among the linking coefficients in stages 20 to 25.
We can obtain linear expressions for 42 of the linking coefficients with the following 33 linking coefficients remaining as parameters.

$$a_{20,i},\; i = 13, 15, 19, \quad a_{21,i},\, i = 13, 15, 19, 20, \quad a_{22,i},\, i = 12, 13, 15, 19, 20, \quad a_{23,i},\, i = 12, 13, 15, 19, 20, 22,$$
$$a_{24,i},\, i = 12, 13, 15, 19, 20, 22, 23, \quad a_{25,i},\, i = 12, 13, 15, 19, 20, 22, 23, 24.$$

Step 5:
A system of equations arising from the 7 column simplifying conditions

$$\sum_{i=j+1}^{25} b_i\, a_{i,j} = b_j\,(1 - c_j), \quad j = 13, 15, 19, 20, 22, 23, 24.$$

can be solved to give linear expressions for $a_{21,13}$, $a_{21,15}$, $a_{21,19}$, $a_{21,20}$, $a_{25,22}$, $a_{25,23}$, $a_{25,24}$.

By means of substitution we obtain linear expressions for 49 linking coefficients with the following 26 linking coefficients remaining as parameters.

$$a_{20,i},\ i = 13, 15, 19, \quad a_{22,i}, i = 12, 13, 15, 19, 20, \quad a_{23,i}, i = 12, 13, 15, 19, 20, 22,$$
$$a_{24,i}, i = 12, 13, 15, 19, 20, 22, 23, \quad a_{25,i}, i = 12, 13, 15, 19, 20.$$

Step 6:
A system of equations arising from the 5 order conditions

$$\sum_{i=3}^{25} b_i \left(\sum_{j=2}^{i-1} a_{i,j}\, c_j^{\,9} \right) = \frac{1}{110}, \qquad \sum_{i=3}^{25} b_i\, c_i \left(\sum_{j=2}^{i-1} a_{i,j}\, c_j^{\,8} \right) = \frac{1}{99}, \qquad \sum_{i=3}^{25} b_i\, c_i^{\,2} \left(\sum_{j=2}^{i-1} a_{i,j}\, c_j^{\,7} \right) = \frac{1}{88},$$

$$\sum_{i=4}^{25} b_i\, c_i^{\,2} \left(\sum_{j=3}^{i-1} a_{i,j} \left(\sum_{k=2}^{j-1} a_{j,k}\, c_k^{\,6} \right) \right) = \frac{1}{616}, \qquad \sum_{i=4}^{25} b_i\, c_i \left(\sum_{j=3}^{i-1} a_{i,j}\, c_j \left(\sum_{k=2}^{j-1} a_{j,k}\, c_k^{\,6} \right) \right) = \frac{1}{693}$$

can be solved to give linear expressions for $a_{23,12}$, $a_{23,22}$, $a_{24,12}$, $a_{24,22}$, $a_{25,12}$.

By means of substitution we obtain linear expressions for 54 linking coefficients with the following 21 linking coefficients remaining as parameters.

$$a_{20,i},\ i = 13, 15, 19, \quad a_{22,i}, i = 12, 13, 15, 19, 20, \quad a_{23,i}, i = 13, 15, 19, 20, \quad a_{24,i}, i = 13, 15, 19, 20, 23, \quad a_{25,i}, i = 13, 15, 19, 20.$$

Step 7:
A system of equations arising from the 4 order conditions

$$\sum_{i=4}^{25} b_i \left(\sum_{j=3}^{i-1} a_{i,j}\, c_j \left(\sum_{k=2}^{j-1} a_{j,k}\, c_k^{\,7} \right) \right) = \frac{1}{880}, \qquad \sum_{i=4}^{25} b_i\, c_i^{\,3} \left(\sum_{j=3}^{i-1} a_{i,j} \left(\sum_{k=2}^{j-1} a_{j,k}\, c_k^{\,5} \right) \right) = \frac{1}{462},$$

$$\sum_{i=4}^{25} b_i\, c_i^{\,2} \left(\sum_{j=3}^{i-1} a_{i,j}\, c_j \left(\sum_{k=2}^{j-1} a_{j,k}\, c_k^{\,5} \right) \right) = \frac{1}{528}, \qquad \sum_{i=4}^{25} b_i\, c_i \left(\sum_{j=3}^{i-1} a_{i,j}\, c_j^{\,2} \left(\sum_{k=2}^{j-1} a_{j,k}\, c_k^{\,5} \right) \right) = \frac{1}{594}$$

can be solved to give linear expressions for $a_{22,12}$, $a_{22,13}$, $a_{23,13}$, $a_{24,23}$.

By means of substitution we obtain linear expressions for 58 linking coefficients in terms of the 17 linking coefficients

$$a_{20,i},\ i = 13, 15, 19, \quad a_{22,i}, i = 15, 19, 20, \quad a_{23,i}, i = 15, 19, 20, \quad a_{24,i}, i = 13, 15, 19, 20, \quad a_{25,i}, i = 13, 15, 19, 20.$$

Step 8:
An equation arising from the order condition

$$\sum_{i=5}^{25} b_i \left(\sum_{j=4}^{i-1} a_{i,j}\, c_j \left(\sum_{k=3}^{j-1} a_{j,k} \left(\sum_{l=2}^{k-1} a_{k,l}\, c_l^{\,6} \right) \right) \right) = \frac{1}{6160}$$

can be solved to give a linear expressions for $a_{23,15}$.
By means of substitution we obtain linear expressions for 59 linking coefficients in terms of the 16 linking coefficients

$$a_{20,i},\ i = 13, 15, 19, \quad a_{22,i}, i = 15, 19, 20, \quad a_{23,i}, i = 19, 20, \quad a_{24,i}, i = 13, 15, 19, 20, \quad a_{25,i}, i = 13, 15, 19, 20.$$

Step 9:
We now specify values for 5 of the linking coefficients in rows 24 and 25, namely $a_{24,15}$, $a_{24,19}$, $a_{25,15}$, $a_{25,19}$, $a_{25,20}$.
By means of substitution we obtain linear expressions for 64 linking coefficients in terms of the 11 linking coefficients

$$a_{20,i},\ i = 13, 15, 19, \quad a_{22,i}, i = 15, 19, 20, \quad a_{23,i}, i = 19, 20, \quad a_{24,i}, i = 13, 20, \quad a_{25,13}.$$

A system of equations in the 11 variable linking coefficients can be constructed from the 11 order conditions

$$\sum_{i=4}^{25} b_i\, c_i \left(\sum_{j=3}^{i-1} a_{i,j} \left(\sum_{k=2}^{j-1} a_{j,k}\, c_k^{\,7}\right)\right) = \frac{1}{792}, \qquad \sum_{i=5}^{25} b_i\, c_i \left(\sum_{j=4}^{i-1} a_{i,j} \left(\sum_{k=3}^{j-1} a_{j,k} \left(\sum_{l=2}^{k-1} a_{k,l}\, c_l^{\,6}\right)\right)\right) = \frac{1}{5544},$$

$$\sum_{i=5}^{25} b_i\, c_i^{\,2} \left(\sum_{j=4}^{i-1} a_{i,j} \left(\sum_{k=3}^{j-1} a_{j,k} \left(\sum_{l=2}^{k-1} a_{k,l}\, c_l^{\,5}\right)\right)\right) = \frac{1}{3696}, \qquad \sum_{i=5}^{25} b_i\, c_i \left(\sum_{j=4}^{i-1} a_{i,j}\, c_j \left(\sum_{k=3}^{j-1} a_{j,k} \left(\sum_{l=2}^{k-1} a_{k,l}\, c_l^{\,5}\right)\right)\right) = \frac{1}{4158},$$

$$\sum_{i=5}^{25} b_i\, c_i \left(\sum_{j=4}^{i-1} a_{i,j} \left(\sum_{k=3}^{j-1} a_{j,k}\, c_k \left(\sum_{l=2}^{k-1} a_{k,l}\, c_l^{\,5}\right)\right)\right) = \frac{1}{4752},$$

$$\sum_{i=6}^{25} b_i\, c_i \left(\sum_{j=5}^{i-1} a_{i,j} \left(\sum_{k=4}^{j-1} a_{j,k} \left(\sum_{l=3}^{k-1} a_{k,l} \left(\sum_{m=2}^{l-1} a_{l,m}\, c_m^{\,5}\right)\right)\right)\right) = \frac{1}{33264},$$

$$\sum_{i=6}^{25} b_i \left(\sum_{j=5}^{i-1} a_{i,j}\, c_j \left(\sum_{k=4}^{j-1} a_{j,k} \left(\sum_{l=3}^{k-1} a_{k,l} \left(\sum_{m=2}^{l-1} a_{l,m}\, c_m^{\,5}\right)\right)\right)\right) = \frac{1}{36960},$$

$$\sum_{i=6}^{25} b_i\, c_i^{\,2} \left(\sum_{j=5}^{i-1} a_{i,j} \left(\sum_{k=4}^{j-1} a_{j,k} \left(\sum_{l=3}^{k-1} a_{k,l} \left(\sum_{m=2}^{l-1} a_{l,m}\, c_m^{\,4}\right)\right)\right)\right) = \frac{1}{18480},$$

$$\sum_{i=7}^{25} b_i\, c_i \left(\sum_{j=6}^{i-1} a_{i,j} \left(\sum_{k=5}^{j-1} a_{j,k} \left(\sum_{l=4}^{k-1} a_{k,l} \left(\sum_{m=3}^{l-1} a_{l,m} \left(\sum_{n=2}^{m-1} a_{m,n}\, c_n^{\,4}\right)\right)\right)\right)\right) = \frac{1}{166320},$$

$$\sum_{i=7}^{25} b_i\, c_i^{\,2} \left(\sum_{j=6}^{i-1} a_{i,j} \left(\sum_{k=5}^{j-1} a_{j,k} \left(\sum_{l=4}^{k-1} a_{k,l} \left(\sum_{m=3}^{l-1} a_{l,m} \left(\sum_{n=2}^{m-1} a_{m,n}\, c_n^{\,3}\right)\right)\right)\right)\right) = \frac{1}{73920},$$

$$\sum_{i=8}^{25} b_i\, c_i \left(\sum_{j=7}^{i-1} a_{i,j} \left(\sum_{k=6}^{j-1} a_{j,k} \left(\sum_{l=5}^{k-1} a_{k,l} \left(\sum_{m=4}^{l-1} a_{l,m} \left(\sum_{n=3}^{m-1} a_{m,n} \left(\sum_{p=2}^{n-1} a_{n,p}\, c_p^{\,3}\right)\right)\right)\right)\right)\right) = \frac{1}{665280}.$$

The resulting equations have respective degrees 2, 2, 2, 2, 2, 3, 2, 2, 3, 2, 3 in the 11 variables and so would be extremely difficult to solve analytically. However, the system can be solved by using the multidimensional version of Newton's method if initial values are provided. In many cases it is sufficient to take the initial value for each of the 11 variables to be zero.
A given order 11 scheme is determined by specifying values for the 19 coefficients

$$c_7,\ c_{13},\ c_{17},\ c_{18},\ c_{19},\ c_{20},\ c_{21},\ c_{22},\ c_{23},\ c_{24},\ a_{18,17},\ a_{19,17},\ a_{19,18},\ b_{25},\ a_{24,15},\ a_{24,19},\ a_{25,15},\ a_{25,19},\ a_{25,20}.$$

In searching for order 11 schemes with reasonable characterstics one may change the values of these 19 parameters incrementally and use the values obtained at a given stage for the 11 coefficients which occur as the variables in the system of non-linear equations to be solved in the last step of the construction of the scheme as starting values for Newton's method in the determination of the coefficients for the next scheme.

Once an order 11 scheme has been constructed, an embedded scheme order 10 scheme can be obtained by adding a 26th row of linking coefficients. We specify that $c_{26} = 1$, $a_{26,i} = 0$ for $i = 2\,..\,10$, $b^*_i = 0$ for $i = 2\,..\,14$. We set $b^*_{25} = 0$ and give a value for b^*_{26} which means that the scheme is essentially a 25 stage scheme.

A system of equations can be constructed using the order 10 quadrature conditions together with the row-sum condition for the additional 26th stage and the stage-order conditions that ensure that this stage has stage-order 7, that is,

$$\sum_{i=1}^{26} b^*_i = 1, \qquad \sum_{i=1}^{26} b^*_i\, c_i^{\,k} = \frac{1}{k+1} \text{ for } k = 1\,..\,9,$$

$$\sum_{j=1}^{25} a_{26,j} = c_{26}, \qquad \sum_{j=2}^{25} a_{26,j}\, c_j^{\,k} = \frac{1}{k+1}\, c_{26}^{\,(k+1)} \text{ for } k = 1\,..\,6.$$

In choosing a value for b^*_{26} we try to ensure that the stability region of the embedded order 10 scheme is compatible with that of the order 11 scheme. We make the order 11 scheme into a 26 stage scheme by specifying that $b_{26} = 0$.
The system of 17 equations can be solved to express the weights b^*_1 and b^*_i for $i = 15$ to 23 in terms of b^*_{24}.
Additionally, the linking coefficients $a_{26,1}$ and $a_{26,i}$, $i = 11$ to 16 can be expressed as linear combinations of the linking coefficients $a_{26,i}$ for $i = 17..25$. The column simplifying conditions

$$\sum_{i=j+1}^{26} b^*_i \, a_{i,j} = b^*_j \,(1 - c_j), \; j = 17..25,$$

can now be used to determine the linking coefficients $a_{26,j}$ for $j = 17..25$ in terms of b^*_{24}.
This enables all the linking to be expressed in tems of b^*_{24}.
Finally, b^*_{24} can be determined by using the single order condition

$$\sum_{i=3}^{26} b^*_i \, c_i \left(\sum_{j=2}^{i-1} a_{i,j} \, c_j^{\,7} \right) = \frac{1}{80}.$$

The stage-orders of stages 3 to 26 of the combined scheme are as follows.

stage	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
stage-order	3	3	3	4	4	4	5	5	5	6	6	6	7	7	7	7	7	7	7	7	7	7	7	7

The principal error norm of an order 11 scheme constructed in the manner described can be calculated using 45 of the 4766 principal error terms. These error terms are given in an abreviated form as follows.

$$b\,c\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,c))))))))) - \frac{1}{43545600}, \qquad b\,c\,(a\,(a\,c\,(a\,(a\,(a\,(a\,(a\,(a\,c)))))))) - \frac{1}{5443200},$$

$$b\,c\,(a\,c\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,c)))))))) - \frac{1}{4838400}, \qquad \frac{1}{2}\left(b\,c^2\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,c)))))))) - \frac{1}{4354560}\right),$$

$$\frac{1}{2}\left(b\,c\,(a\,c^2\,(a\,(a\,(a\,(a\,(a\,(a\,c))))))) - \frac{1}{604800}\right), \qquad \frac{1}{2}\left(b\,c^2\,(a\,c\,(a\,(a\,(a\,(a\,(a\,(a\,c))))))) - \frac{1}{544320}\right),$$

$$\frac{1}{6}\left(b\,c\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,c^3))))))) - \frac{1}{7257600}\right), \qquad \frac{1}{6}\left(b\,c^3\,(a\,(a\,(a\,(a\,(a\,(a\,(a\,c))))))) - \frac{1}{483840}\right),$$

$$\frac{1}{6}\left(b\,c\,(a\,(a\,c\,(a\,(a\,(a\,(a\,c^3)))))) - \frac{1}{907200}\right), \qquad \frac{1}{6}\left(b\,c\,(a\,c\,(a\,(a\,(a\,(a\,(a\,c^3)))))) - \frac{1}{806400}\right),$$

$$\frac{1}{12}\left(b\,c^2\,(a\,(a\,(a\,(a\,(a\,(a\,c^3)))))) - \frac{1}{725760}\right), \qquad \frac{1}{24}\left(b\,c\,(a\,(a\,(a\,(a\,(a\,(a\,c^4)))))) - \frac{1}{1814400}\right),$$

$$\frac{1}{24}\left(b\,c^4\,(a\,(a\,(a\,(a\,(a\,(a\,c)))))) - \frac{1}{60480}\right), \qquad \frac{1}{12}\left(b\,c\,(a\,c^2\,(a\,(a\,(a\,(a\,c^3))))) - \frac{1}{100800}\right),$$

$$\frac{1}{12}\left(b\,c^2\,(a\,c\,(a\,(a\,(a\,(a\,c^3))))) - \frac{1}{90720}\right), \qquad \frac{1}{36}\left(b\,c^3\,(a\,(a\,(a\,(a\,(a\,c^3))))) - \frac{1}{80640}\right),$$

$$\frac{1}{24}\left(b\,c\,(a\,(a\,c\,(a\,(a\,(a\,c^4))))) - \frac{1}{226800}\right), \qquad \frac{1}{24}\left(b\,c\,(a\,c\,(a\,(a\,(a\,(a\,c^4))))) - \frac{1}{201600}\right),$$

$$\frac{1}{48}\left(b\,c^2\,(a\,(a\,(a\,(a\,(a\,c^4))))) - \frac{1}{181440}\right), \qquad \frac{1}{120}\left(b\,c\,(a\,(a\,(a\,(a\,(a\,c^5))))) - \frac{1}{362880}\right),$$

$$\frac{1}{48}\left(b\,c\,(a\,c^2\,(a\,(a\,(a\,c^4))))-\frac{1}{25200}\right),\qquad \frac{1}{48}\left(b\,c^2\,(a\,c\,(a\,(a\,(a\,c^4))))-\frac{1}{22680}\right),$$

$$\frac{1}{144}\left(b\,c^3\,(a\,(a\,(a\,(a\,c^4))))-\frac{1}{20160}\right),\qquad \frac{1}{144}\left(b\,c^4\,(a\,(a\,(a\,(a\,c^3))))-\frac{1}{10080}\right),$$

$$\frac{1}{120}\left(b\,c\,(a\,(a\,c\,(a\,(a\,c^5))))-\frac{1}{45360}\right),\qquad \frac{1}{120}\left(b\,c\,(a\,c\,(a\,(a\,(a\,c^5))))-\frac{1}{40320}\right),$$

$$\frac{1}{240}\left(b\,c^2\,(a\,(a\,(a\,(a\,c^5))))-\frac{1}{36288}\right),\qquad \frac{1}{720}\left(b\,c\,(a\,(a\,(a\,(a\,c^6))))-\frac{1}{60480}\right),$$

$$\frac{1}{576}\left(b\,c^4\,(a\,(a\,(a\,c^4)))-\frac{1}{2520}\right),\qquad \frac{1}{240}\left(b\,c\,(a\,c^2\,(a\,(a\,c^5)))-\frac{1}{5040}\right),\qquad \frac{1}{240}\left(b\,c^2\,(a\,c\,(a\,(a\,c^5)))-\frac{1}{4536}\right),$$

$$\frac{1}{720}\left(b\,c^3\,(a\,(a\,(a\,c^5)))-\frac{1}{4032}\right),\qquad \frac{1}{720}\left(b\,c\,(a\,c\,(a\,(a\,c^6)))-\frac{1}{6720}\right),\qquad \frac{1}{1440}\left(b\,c^2\,(a\,(a\,(a\,c^6)))-\frac{1}{6048}\right),$$

$$\frac{1}{5040}\left(b\,c\,(a\,(a\,(a\,c^7)))-\frac{1}{8640}\right),\qquad \frac{1}{2880}\left(b\,c^4\,(a\,(a\,c^5))-\frac{1}{504}\right),\qquad \frac{1}{4320}\left(b\,c^3\,(a\,(a\,c^6))-\frac{1}{672}\right),$$

$$\frac{1}{5040}\left(b\,c\,(a\,c\,(a\,c^7))-\frac{1}{960}\right),\qquad \frac{1}{10080}\left(b\,c^2\,(a\,(a\,c^7))-\frac{1}{864}\right),\qquad \frac{1}{40320}\left(b\,c\,(a\,(a\,c^8))-\frac{1}{1080}\right),$$

$$\frac{1}{30240}\left(b\,c^3\,(a\,c^7)-\frac{1}{96}\right),\qquad \frac{1}{80640}\left(b\,c^2\,(a\,c^8)-\frac{1}{108}\right),\qquad \frac{1}{362880}\left(b\,c\,(a\,c^9)-\frac{1}{120}\right),$$

$$\frac{1}{3628800}\left(b\,(a\,c^{10})-\frac{1}{132}\right),\qquad \frac{1}{39916800}\left(b\,c^{11}-\frac{1}{12}\right).$$

For example, $\frac{1}{240}\left(b\,c^2\,(a\,c\,(a\,(a\,c^5)))-\frac{1}{4536}\right)$ is an abreviation for

$$\frac{1}{240}\left(\left(\sum_{i=2}^{25} b_i\,c_i^2\left(\sum_{j=3}^{i-1} a_{i,j}\,c_j\left(\sum_{k=4}^{j-1} a_{j,k}\left(\sum_{l=5}^{k-1} a_{k,l}\,c_l^5\right)\right)\right)\right)-\frac{1}{4536}\right).$$

The principal error norm can be calculated as $\sqrt{\sum_{i=1}^{45} d_i\,e_i^2}$, where e_i is the value of the ith error term above and d_i is the ith member of the sequence
2, 2, 2, 6, 4, 6, 22, 30, 22, 22, 66, 156, 282, 44, 66, 330, 156, 156, 468, 2052, 312, 468, 2340, 3102, 2052, 2052, 6156, 39432, 21996, 4104, 6156, 30780, 39432, 118296, 950112, 289332, 591480, 950112, 2850336, 28905348, 14251680, 86716044, 1028434308, 32474566307, 1.

The following sets of parameter values determine schemes with the given characteristics.

1. **Scheme A**.

$$c_7=\frac{190}{1457},\ c_{13}=\frac{1849}{2297},\ c_{17}=\frac{1406}{1419},\ c_{18}=\frac{478}{5311},\ c_{19}=\frac{2587}{6417},\ c_{20}=\frac{2381}{9812},\ c_{21}=\frac{2063}{2862},\ c_{22}=\frac{1256}{6281},\ c_{23}=\frac{8729}{9055},\ c_{24}=\frac{3173}{4267},\ c_{25}=1,$$

$$a_{18,\,17}=\frac{153}{1583},\ a_{19,\,17}=\frac{526}{983},\ a_{19,\,18}=\frac{1975}{9206},\ b_{25}=-\frac{1356}{1471},$$

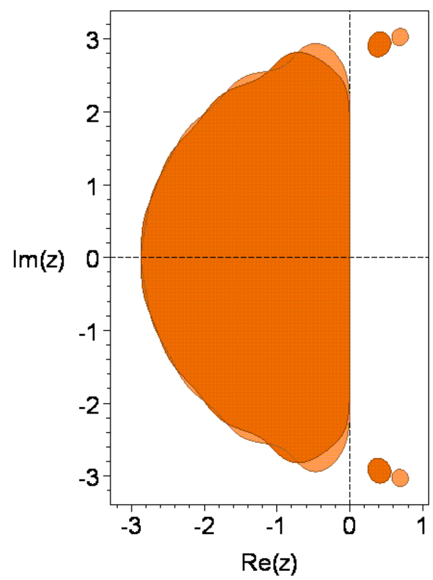
$$a_{24,\,15}=-\frac{3523}{2657},\ a_{24,\,19}=-\frac{3811}{4873},\ a_{25,\,15}=-\frac{4715}{3458},\ a_{25,\,19}=-\frac{3658}{2551},\ a_{25,\,20}=-\frac{2059}{7759}.$$

The order 11 scheme has a principal error norm $0.16737047 \times 10^{(-6)}$ and the real stability interval $[-2.86308, 0]$.

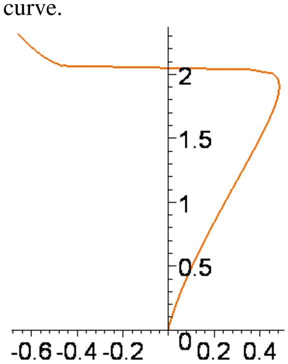
The weight that determines the order 10 embedded scheme is $b^*_{26} = \frac{12}{29}$.

The order 10 scheme has a principal error norm $0.52127319 \times 10^{(-6)}$ and the real stability interval $[-2.86322, 0]$.

The stability regions of the two schemes are shown in the following picture in which the stability region of the order 10 scheme is given the darker shade.



The following picture shows the result of distorting the boundary curve of the stability region of the order 11 scheme horizontally by taking the 11th root of the real part of points along the curve.



The stability region intersects the nonnegative imaginary axis in the interval $[0, 2.03877\,i]$.

The maximum magnitude of the linking coefficients is 17.134789.
The 2-norm of the linking coefficients is 34.757959.

2. **Scheme B**.

$$c_7 = \frac{2213}{7686},\ c_{13} = \frac{4821}{6055},\ c_{17} = \frac{2729}{28575},\ c_{18} = \frac{4795}{9144},\ c_{19} = \frac{7748}{9605},\ c_{20} = \frac{2744}{17139},\ c_{21} = \frac{749}{3641},\ c_{22} = \frac{3415}{4681},\ c_{23} = \frac{5642}{6787},\ c_{24} = \frac{4908}{4997},\ c_{25} = 1,$$

$$a_{18,17} = -\frac{962}{41393},\ a_{19,17} = -\frac{1843}{17387},\ a_{19,18} = \frac{17095}{6468},\ b_{25} = \frac{771}{12268},$$

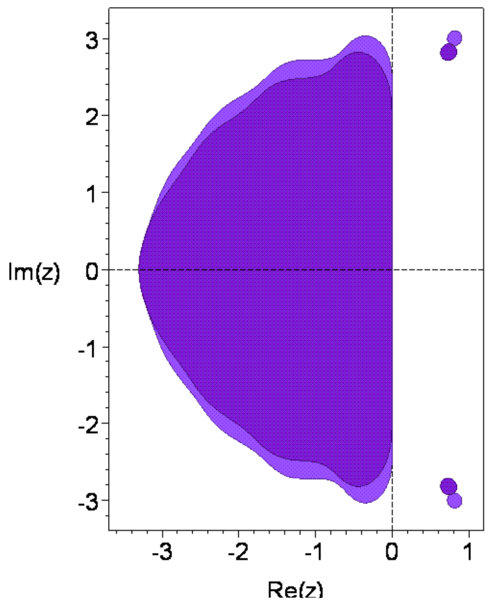
$$a_{24,15} = -\frac{1244}{5583},\ a_{24,19} = \frac{3749}{44863},\ a_{25,15} = -\frac{3207}{34774},\ a_{25,19} = -\frac{6637}{7368},\ a_{25,20} = -\frac{17075}{4919}.$$

The order 11 scheme has a principal error norm $0.90151732 \times 10^{(-7)}$ and the real stability interval $[-3.30296, 0]$.

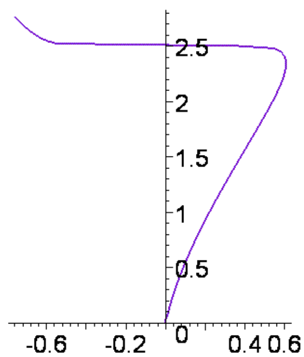
The weight that determines the order 10 embedded scheme is $b^*_{26} = \frac{6}{143}$.

The order 10 scheme has a principal error norm $0.27684033 \times 10^{(-6)}$ and the real stability interval $[-3.30285, 0]$.

The stability regions of the two schemes are shown in the following picture in which the stability region of the order 10 scheme is given the darker shade.



The following picture shows the result of distorting the boundary curve of the stability region of the order 11 scheme horizontally by taking the 11th root of the real part of points along the curve.



The stability region intersects the nonnegative imaginary axis in the interval $[0, 2.503825\, i]$.

The maximum magnitude of the linking coefficients is 65.498252.
The 2-norm of the linking coefficients is 178.17428.

Version: 10 January 2015, Peter Stone