

**A 21 stage combined order 9 and 10 Runge-Kutta scheme**

The combined 10(9) scheme is based on an 18 stage order 10 scheme for which the nodes  $c_{12}, c_{13}, c_{14}, c_{15}$  are zeros of the derivative

$P'_5(x) = \frac{d}{dx} P_5(x)$  of the Legendre polynomial  $P_5(x)$  mapped linearly from the interval  $[-1, 1]$  to the interval  $[0, 1]$ . Three additional stages are required for the embedded order 9 scheme.

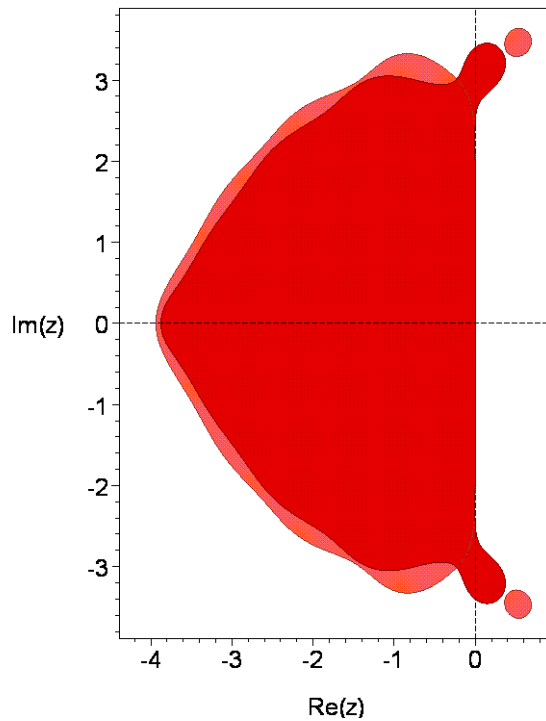
The principal error norm of the order 10 scheme, that is, the 2-norm of the principal error terms is  $0.2797129535 \times 10^{(-6)}$ .

The principal error norm of the embedded order 9 scheme is  $0.1228271247 \times 10^{(-4)}$ .

The maximum magnitude of the linking coefficients is 9.251611659.

The 2-norm of the linking coefficients is 23.40459060.

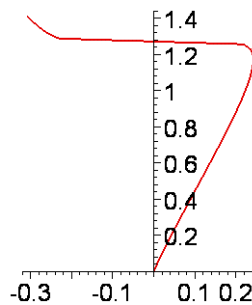
The stability regions for the two schemes are shown in the following picture.



The stability region of the order 9 scheme appears in the darker shade.

The real stability intervals of the order 10 and 9 schemes are respectively  $[-3.93592, 0]$  and  $[-3.87594, 0]$ .

The following picture shows the result of distorting the boundary curve of the stability region of the order 10 scheme horizontally by taking the 11th root of the real part of points along the curve.



The stability region intersects the nonnegative imaginary axis in the interval  $[0, 1.27032 i]$ .



a[10,1]=.7482991442860577981796457881779403766541653418031507171270087634017352737394460318694e-1,  
a[10,2]=0.,  
a[10,3]=0.,  
a[10,4]=0.,  
a[10,5]=0.,  
a[10,6]=.2146242453344611455132203077334015805531676911974595841457907964297528509081510410051,  
a[10,7]=.8441587281532899509438233442785162155862135776522459649160665852361674327325848240811e-1,  
a[10,8]=-.4341731449884088314258244505051059128188086758915162983342160928715665785520723965808e-1,  
a[10,9]=-.2127342714776132897101692419479541242234800553596728652351233239481340365156808675246,  
a[11,1]=.8105319135213137982449097494946605318699679534895426804764531260518855925045066929146e-1,  
a[11,2]=0.,  
a[11,3]=0.,  
a[11,4]=0.,  
a[11,5]=0.,  
a[11,6]=.3072347902786314723782750840848327000813657490129430709594567604719975889556859980727,  
a[11,7]=.3221330653537545735345358704131438377363578220994411486998770098713527182328054424358,  
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a[11,10]=.4253112033195020746887966804979253112033195020746887966804979253112033195020746887967e-1,  
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a[12,3]=0.,  
a[12,4]=0.,  
a[12,5]=0.,  
a[12,6]=0.,  
a[12,7]=0.,  
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a[12,11]=.2983421865355255214984303827590516572717567566404957862226759633762007215101150163603,  
a[13,1]=.4502616373258378089571585266943067264191105647558891070426642561293135450072038273401e-1,  
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a[13,3]=0.,  
a[13,4]=0.,  
a[13,5]=0.,  
a[13,6]=0.,  
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a[14,6]=0.,  
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a[15,2]=0.,  
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a[15,5]=0.,  
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a[16,14]=.8992773696348355846430438306111181223414632598285854300924423251352733205187087732678e-1,  
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a[17,14]=- .9161854401769482236671414092387917470686251714192236693920061138984202381209109248553e-1,  
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a[18,2]=0.,  
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a[18,4]=0.,  
a[18,5]=0.,  
a[18,6]=0.,  
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a[18,10]=8.837479166321531557132713091438078349012950420883408996400023695702859437693776872331,  
a[18,11]=-3.39531550492577522168778009491669502852993530906830134150026021994027862806447239008,  
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a[18,14]=.3189152692455334369024560213486753019540464785641163242047782111839399471147176681561,  
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a[18,16]=- .6051983612219277832241707671295607127814820499715293613761402732652780120810041653591,

a[18,17]=.3334525350307787459202631378414806560287636505658634784117511174230383993073398823363,  
a[19,1]=.323552803925102999936248322986404255196342688922119433461863555966595757394320390300e-2,  
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a[19,4]=0.,  
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a[19,10]=-3.105719709275240232197214063392261855105931466642070666791231923653448375868112099569,  
a[19,11]=.3735792316281636099494639652919331536371486831512926431188239045324084394660410921473,  
a[19,12]=.6457904134285096860345874008864280380634150697829300595746931514884010154958497333426,  
a[19,13]=3.395903253925594454341688858064139134191953266336726319040842498849961167762236495090,  
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a[19,15]=0.,  
a[19,16]=0.,  
a[19,17]=0.,  
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a[20,1]=-6.313873841589129167947486243496550109203559261800955082701590867138272697422089895059,  
a[20,2]=0.,  
a[20,3]=0.,  
a[20,4]=0.,  
a[20,5]=0.,  
a[20,6]=.4245913759831255017814404450922958609173886480136563894001880512345162012210239508743,  
a[20,7]=-3.225053843197273548777516463708481200346658724722450299105430801813107526895723872045,  
a[20,8]=1.127502503388317521779090661463491151704003378589850048165433855949276333896277437886,  
a[20,9]=2.556225387908715149988783698446576372749976236321998900224730583021523887262450021065,  
a[20,10]=-3.503563466548935411694348974746889683327113799964372600060342733804344642796483188604,  
a[20,11]=4.131826065261408908935833333182507975083266891655828531552211939349788349670783690692,  
a[20,12]=-4.018894760358292040781221823531782447471894169283419727267928057577614981824154265376,  
a[20,13]=6.267699444743103818422725961814587664681213215345987816698108273685810983812012368735,  
a[20,14]=-5.509808114010958273422783517488140338178549343990637298838087198992931763185528874335,  
a[20,15]=0.,  
a[20,16]=0.,  
a[20,17]=0.,  
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a[20,19]=-1.777964511620161155517759861913836649251970815377919821453003303432728158285424266288,  
a[21,1]=-2.144917830410915206075361693219001310298605493627943041798948924583064794686910332627,  
a[21,2]=0.,  
a[21,3]=0.,  
a[21,4]=0.,  
a[21,5]=0.,  
a[21,6]=.6471532422526927707543492250217164673191514616930459796286204351750245480148649201690,  
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a[21,10]=3.235294117647058823529411764705882352941176470588235294117647058823529411764705882353,  
a[21,11]=2.040979384582947465323995605709526776520452277473976795756456286488725812245266023419,  
a[21,12]=.3824161301209826666172062398765295849765155410646081500527496679929903077476792042151e-1,  
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a[21,14]=.2,  
a[21,15]=0.,  
a[21,16]=0.,  
a[21,17]=0.,  
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a[21,20]=-2.441650463286808031195981858683132078589910054807668026691405691587469463237595649692,

