

A 22 stage combined order 9 and 10 Runge-Kutta scheme

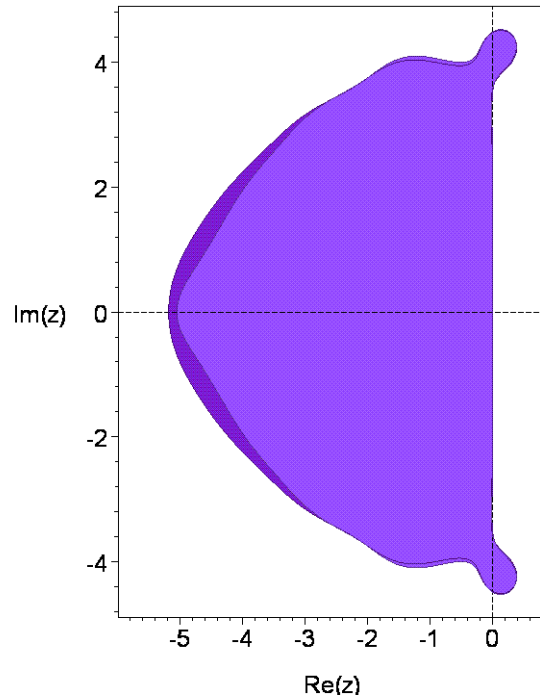
The principal error norm of the order 10 scheme, that is, the 2-norm of the principal error terms is: $0.6001588154 \times 10^{(-7)}$.

The principal error norm of the embedded order 9 scheme is: $0.3141270351 \times 10^{(-6)}$.

The maximum magnitude of the linking coefficients is 16.19434756.

The 2-norm of the linking coefficients is 43.78037143.

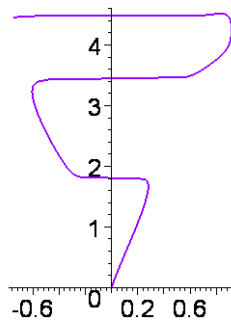
The stability regions for the two schemes are shown in the following picture.



The stability region of the order 9 scheme appears in the darker shade.

The real stability intervals of the order 10 and 9 schemes are respectively $[-5.0510, 0]$ and $[-5.18345, 0]$.

The following picture shows the result of distorting the boundary curve of the stability region of the order 10 scheme horizontally by taking the 11th root of the real part of points along the curve.



The intersection of the stability region with the nonnegative imaginary axis is $[0, 1.8137] \cup [3.43665, 4.4798]$.

The coefficients correct to 85 digits are as follows.

c[2]=.2293083549457312689461548072529765259204832314037348446266278165172449932015501667796,
c[3]=.3439625324185969034192322108794647888807248471056022669399417247758674898023252501694,
c[4]=.5159437986278953551288483163191971833210872706584034004099125871638012347034878752541,
c[5]=.8173332669635780826327970587533285096636711366849365378384465696826113517718522276536e-1,
c[6]=.6712589048950083922190443106934996695565611421871681735957852408854076821602003501539,
c[7]=.1248073959938366718027734976887519260400616332819722650231124807395993836671802773498,
c[8]=.4341318961120407008917038791985528231420341268414576292745192689471087220467385627037,
c[9]=.7121857431968124380016436558042033039675486508033174720192162841813111583282471263570,
c[10]=.1894273173120404448775779085268308117473857559589161668260437288552562870480031138267,
c[11]=.4943589743589743589743589743589743589743589743589743589743589743589743589743589743590,
c[12]=.6402983958815653776707324111702903090135837808950884015348862059924236276588471145808,
c[13]=.7415384615384615384615384615384615384615384615384615384615384615384615384615384615385,
c[14]=.3868904876099120703437250199840127897681854516386890487609912070343725019984012789768,
c[15]=.1073457023854600530102234002271866717152593714502082544490723210904960242332449829610,
c[16]=.8734655335221907459867799811142587346553352219074598677998111425873465533522190745987,
c[17]=.9613913043478260869565217391304347826086956521739130434782608695652173913043478260870,
c[18]=.3302411873840445269016697588126159554730983302411873840445269016697588126159554730983,
c[19]=.6261682242990654205607476635514018691588785046728971962616822429906542056074766355140,
c[20]=.9688768606224627875507442489851150202976995940460081190798376184032476319350473612991,
c[21]=1.,
c[22]=1.,

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a[6,5]=.4344572904565130973112339136784623070387508809949911668720456530127076855762718530731,
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a[8,6]=.1946153342786839679924231335328732633164924971471027512494606124602981408981459906505e-1,
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a[9,1]=.2262406244004235251383910367819960619900619101468899345768463599467029415667939082119e-1,
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a[9,8]=.3232041548618899946186701216472392550350325503523177070221358669708165932473615642397,
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a[10,3]=0.,
a[10,4]=0.,
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a[20,15]=4.171045595589845819485525150712360752485079003998405745013561993718589220100438923028,
a[20,16]=.9990905309898352292295275297264516654429418039650542911786136547493965590499601769126e-1,
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a[21,1]=- 1.528519298068126098144204509833248467042954507860448447702169947825908788929223634505,
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a[21,20]=- .3670931101351098184272080302065037184507141341491422689734624521116681269896537822107,
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a[22,5]=0.,
a[22,6]=0.,
a[22,7]=0.,
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a[22,9]=1.384131469905738741066740678902026070516299430990878830494738699775177737814761227256,
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a[22,11]=-13.36642266759188973890425795640398780212950999231064113094313563817312387135020686967,
a[22,12]=-7.959850079525369929492040200058073573326978171708351849603452835626476359514116129896,
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b[17]=-1.591305661016181373991179300401077570287845184292515945253486940529885689237606038585,
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b[20]=1.680268409490573686667682591632815241750925642446928405882861214617593653134323652081,
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b*[21]=0.,
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