Construction of 21 stage combined order 9 and 10 Runge-Kutta schemes

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The initial steps in the construction of this scheme involve ideas from a 1975 scheme of A.R.Curtis.

See: High-order Explicit Runge-Kutta Formulae, Their uses, and Limitations, A.R.Curtis,
Journal of the Institute of Mathematics and its Applications (1975) 16, pages 35 to 55. J. Inst. Maths Applics Vol. **16** (1975), 35-55.

In a departure from the construction of Curtis' scheme we do not specify that $c_{11} = c_{15},\ c_{10} = c_{13}$.
We also remove the requirements that $a_{3,\,1} = a_{3,\,2}, c_5 = a_{5,\,1}, a_{11,\,10} = 0$ and $a_{17,\,10} = a_{18,\,10}$.

This allows for the employment of the six parameters $c_2,\ c_5,\ c_{10},\ c_{11},\ a_{11,\,10},\ a_{14,\,10}$ in the description of a scheme. These parameters can then be varied in an attempt to obtain a scheme with desirable characteristics.

There is an additional benefit in that it becomes possible to construct an embedded order 9 scheme by adding 3 additional stages, with additional parameters which can also be varied.

We first indicate how to construct an 18 stage order 10 Runge-Kutta scheme.

Step 1:
We use the quadrature conditions to determine the weights and some of the nodes assuming that the following symmetry conditions hold.

$$b_{13} = b_{14} + b_{17},\ b_{14} = b_{17},\ b_{12} = b_{16},\ b_{15} = b_{12} + b_{16},$$

$$c_{12} = c_{16},\ c_{14} = c_{17},\ c_{10} + c_{14} = 1,\ c_{11} + c_{12} = 1,$$

Assuming also that $c_{18} = 1$ and that $b_j = 0,\ \ j = 2, 3,\ \ldots, 11$, we obtain the following values.

$$c_{12} = \frac{1}{2} + \frac{\sqrt{147 - 42\sqrt{7}}}{42},\quad c_{13} = \frac{1}{2} - \frac{\sqrt{147 + 42\sqrt{7}}}{42},\quad c_{14} = \frac{1}{2} + \frac{\sqrt{147 + 42\sqrt{7}}}{42},\quad c_{15} = \frac{1}{2} - \frac{\sqrt{147 - 42\sqrt{7}}}{42},$$

$$c_{16} = \frac{1}{2} + \frac{\sqrt{147 - 42\sqrt{7}}}{42},\quad c_{17} = \frac{1}{2} + \frac{\sqrt{147 + 42\sqrt{7}}}{42},$$

$$b_1 = \frac{1}{30},\ b_{12} = \frac{7}{60} + \frac{\sqrt{7}}{120},\ b_{13} = \frac{7}{30} - \frac{\sqrt{7}}{60},\ b_{14} = \frac{7}{60} - \frac{\sqrt{7}}{120},\ b_{15} = \frac{7}{30} + \frac{\sqrt{7}}{60},\ b_{16} = \frac{7}{60} + \frac{\sqrt{7}}{120},\ b_{17} = \frac{7}{60} - \frac{\sqrt{7}}{120},\ b_{18} = \frac{1}{30}.$$

The nodes:

$$c_{12} = \frac{1}{2} + \frac{\sqrt{147 - 42\sqrt{7}}}{42},\ c_{13} = \frac{1}{2} - \frac{\sqrt{147 + 42\sqrt{7}}}{42},\ c_{14} = \frac{1}{2} + \frac{\sqrt{147 + 42\sqrt{7}}}{42},\ c_{15} = \frac{1}{2} - \frac{\sqrt{147 - 42\sqrt{7}}}{42}$$

are the zeros of the derivative $P'_5(x) = \frac{d}{dx}P_5(x)$ of the **Legendre polynomial** $P_5(x)$ mapped linearly from the interval $[-1, 1]$ to the interval $[0, 1]$. They provide nodes for **Gauss-Lobatto integration** on the interval $[0, 1]$.

We specify the following stage-orders for stages 3 to 18.

stage	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
stage-order	2	3	3	4	4	5	5	5	5	6	6	6	6	6	6	6

Thus we have the following stage-order conditions involving the nodes c_i and linking coefficients $a_{i,j}$.

$$\sum_{j=1}^{i-1} a_{i,j} = c_i \text{ for } i = 3..18, \quad \sum_{j=2}^{i-1} a_{i,j}\, c_j = \frac{c_i^2}{2} \text{ for } i = 3..18, \quad \sum_{j=2}^{i-1} a_{i,j}\, c_j^2 = \frac{c_i^3}{3} \text{ for } i = 4..18,$$

$$\sum_{j=2}^{i-1} a_{i,j}\, c_j^3 = \frac{c_i^4}{4} \text{ for } i = 6..18, \quad \sum_{j=2}^{i-1} a_{i,j}\, c_j^4 = \frac{c_i^5}{5} \text{ for } i = 8..16, \quad \sum_{j=2}^{i-1} a_{i,j}\, c_j^5 = \frac{c_i^6}{6} \text{ for } i = 12..18.$$

We also have zero linking coefficients as indicated by the following tableau.

$$\begin{bmatrix}
c_2 & a_{2,1} \\
c_3 & a_{3,1} & a_{3,2} \\
c_4 & a_{4,1} & 0 & a_{4,3} \\
c_5 & a_{5,1} & 0 & a_{5,3} & a_{5,4} \\
c_6 & a_{6,1} & 0 & 0 & a_{6,4} & a_{6,5} \\
c_7 & a_{7,1} & 0 & 0 & 0 & a_{7,5} & a_{7,6} \\
c_8 & a_{8,1} & 0 & 0 & 0 & a_{8,5} & a_{8,6} & a_{8,7} \\
c_9 & a_{9,1} & 0 & 0 & 0 & 0 & a_{9,6} & a_{9,7} & a_{9,8} \\
c_{10} & a_{10,1} & 0 & 0 & 0 & 0 & a_{10,6} & a_{10,7} & a_{10,8} & a_{10,9} \\
c_{11} & a_{11,1} & 0 & 0 & 0 & 0 & a_{11,6} & a_{11,7} & a_{11,8} & a_{11,9} & a_{11,10} \\
c_{12} & a_{12,1} & 0 & 0 & 0 & 0 & 0 & 0 & a_{12,8} & a_{12,9} & a_{12,10} & a_{12,11} \\
c_{13} & a_{13,1} & 0 & 0 & 0 & 0 & 0 & 0 & a_{13,8} & a_{13,9} & a_{13,10} & a_{13,11} & a_{13,12} \\
c_{14} & a_{14,1} & 0 & 0 & 0 & 0 & 0 & 0 & a_{14,8} & a_{14,9} & a_{14,10} & a_{14,11} & a_{14,12} & a_{14,13} \\
c_{15} & a_{15,1} & 0 & 0 & 0 & 0 & 0 & 0 & a_{15,8} & a_{15,9} & a_{15,10} & a_{15,11} & a_{15,12} & a_{15,13} & a_{15,14} \\
c_{16} & a_{16,1} & 0 & 0 & 0 & 0 & 0 & 0 & a_{16,8} & a_{16,9} & a_{16,10} & a_{16,11} & a_{16,12} & a_{16,13} & a_{16,14} & a_{16,15} \\
c_{17} & a_{17,1} & 0 & 0 & 0 & 0 & 0 & 0 & a_{17,8} & a_{17,9} & a_{17,10} & a_{17,11} & a_{17,12} & a_{17,13} & a_{17,14} & a_{17,15} & a_{17,16} \\
c_{18} & a_{18,1} & 0 & 0 & 0 & 0 & 0 & 0 & a_{18,8} & a_{18,9} & a_{18,10} & a_{18,11} & a_{18,12} & a_{18,13} & a_{18,14} & a_{18,15} & a_{18,16} & a_{18,17}
\end{bmatrix}$$

We can use this information to obtain certain relations involving the nodes c_3 to c_{12}.

$$\int_0^{c_4} x\,(x - c_3)\, dx = 0 \quad \text{or} \quad c_3 = \frac{2}{3} c_4 \quad \text{------------- (i)}$$

$$\int_0^{c_6} x\,(x - c_4)\,(x - c_5)\, dx = 0 \quad \text{or} \quad c_4 = \frac{c_6\,(3\,c_6 - 4\,c_5)}{2\,(2\,c_6 - 3\,c_5)} \quad \text{------------- (iii)}$$

$$\int_0^{c_8} x\,(x - c_6)\,(x - c_7)\, dx = 0 \quad \text{or} \quad c_6 = \frac{c_8\,(3\,c_8 - 4\,c_7)}{2\,(2\,c_8 - 3\,c_7)} \quad \text{------------- (iii)}$$

$$\int_0^{c_8} x^2\,(x - c_6)\,(x - c_7)\, dx = 0 \quad \text{or} \quad c_6 = \frac{c_8\,(4\,c_8 - 5\,c_7)}{2\,(3\,c_8 - 4\,c_7)} \quad \text{------------- (iv)}$$

$$\int_0^{c_9} x\,(x - c_6)\,(x - c_7)\,(x - c_8)\, dx = 0 \quad \text{or} \quad c_6 = \frac{(20\,c_8\,c_7 - 15\,c_9\,c_8 + 12\,c_9^2 - 15\,c_9\,c_7)\,c_9}{5\,(6\,c_8\,c_7 - 4\,c_9\,c_8 + 3\,c_9^2 - 4\,c_9\,c_7)} \quad \text{------------- (v)}$$

$$\int_0^{c_{12}} x\,(x-c_8)\,(x-c_9)\,(x-c_{10})\,(x-c_{11})\,dx = 0 \quad \text{or}$$

$$c_8 = \frac{(20\,c_{11}\,c_{10}\,c_9 - 15\,c_{11}\,c_{10}\,c_{12} + 12\,c_{11}\,c_{12}^2 - 15\,c_{11}\,c_9\,c_{12} + 12\,c_{10}\,c_{12}^2 - 15\,c_{10}\,c_9\,c_{12} - 10\,c_{12}^3 + 12\,c_9\,c_{12}^2)\,c_{12}}{30\,c_{11}\,c_{10}\,c_9 - 20\,c_{11}\,c_{10}\,c_{12} + 15\,c_{11}\,c_{12}^2 - 20\,c_{11}\,c_9\,c_{12} + 15\,c_{10}\,c_{12}^2 - 20\,c_{10}\,c_9\,c_{12} - 12\,c_{12}^3 + 15\,c_9\,c_{12}^2} \quad \text{------------- (vi)}$$

If we specify that $c_7 < c_6$, equations (iii) and (iv) imply that $c_6 = \left(\frac{3}{5} + \frac{\sqrt{6}}{10}\right) c_8$ and $c_7 = \left(\frac{3}{5} - \frac{\sqrt{6}}{10}\right) c_8$.

Specifying values for the nodes c_2, c_5, c_{10} and c_{11} together with some restriction, such as specifing that the sum of the nodes c_6, c_7, c_8, c_9 is a maximum, allows all the remaining nodes to be calculated.
Giving a value for $a_{11,\,10}$ allows all the linking coefficients in the first 13 rows to be obtained by using the stage-order conditions that apply to these rows.

Step 2:
The stage-order conditions for rows 14 to 18 allow various linear relations among the linking coefficients in stages 14 to 18 to be obtained. We can obtain linear expressions for 30 of the linking coefficients with the following 15 linking coefficients remaining as parameters.

$$a_{14,\,10},\quad a_{15,\,i},\ i = 10, 11,\quad a_{16,\,i},\ i = 10, 11, 12,\quad a_{17,\,i},\ i = 10, 11, 12, 13,\quad a_{18,\,i},\ i = 10, 11, 12, 13, 14.$$

Step 3:
A system of equations arising from the 4 column simplifying conditions

$$\sum_{i=j+1}^{18} b_i\,a_{i,j} = b_j\,(1 - c_j), \quad j = 11, 15, 16, 17$$

can be solved to give linear expressions for $a_{16,\,12}$, $a_{18,\,11}$, $a_{18,\,12}$, $a_{18,\,14}$.

By means of substitution we obtain linear expressions for 34 linking coefficients with the following 11 linking coefficients remaining as parameters.

$$a_{14,\,10},\quad a_{15,\,i},\ i = 10, 11,\quad a_{16,\,i},\ i = 10, 11,\quad a_{17,\,i},\ i = 10, 11, 12, 13,\quad a_{18,\,i},\ i = 10, 13.$$

Step 4:
A system of equations arising from the 6 equations

$$\sum_{i=j+1}^{18} b_i\,c_i\,a_{i,j} = 0, \quad j = 9, 10, 11 \quad \text{and} \quad \sum_{i=j+1}^{18} b_i\,c_i^2\,a_{i,j} = 0, \quad j = 9, 10, 11.$$

can be solved to give linear expressions for $a_{16,\,11}$, $a_{17,\,10}$, $a_{17,\,11}$, $a_{17,\,13}$, $a_{18,\,10}$, $a_{18,\,13}$.

By means of substitution we obtain linear expressions for 40 linking coefficients with the following 5 linking coefficients remaining as parameters.

$$a_{14,\,10},\ a_{15,\,10},\ a_{15,\,11},\ a_{16,\,10},\ a_{17,\,12}.$$

Step 5:
We now consider the single column simplifying condition (which was omitted at an earlier stage).

$$\sum_{i=14}^{18} b_i\,a_{i,\,13} = b_{13}\,(1 - c_{13}),$$

that is,

$$b_{14}\,a_{14,\,13} + b_{15}\,a_{15,\,13} + b_{16}\,a_{16,\,13} + b_{17}\,a_{17,\,13} + b_{18}\,a_{18,\,13} = b_{13}\,(1 - c_{13}).$$

The equation arising from this condition can be solved for to obtain an expression for $a_{16,10}$ which results in expressions for 41 linking coefficients with the 4 linking coefficients

$$a_{14,10},\ a_{15,10},\ a_{15,11},\ a_{17,12},$$

remaining as parameters.

Step 6:
At this stage we specify a value for a[14,10]. A system of equations arising from the 3 order conditions

$$\sum_{i=4}^{18} b_i\,c_i\left(\sum_{j=3}^{i-1} a_{i,j}\left(\sum_{k=2}^{j-1} a_{j,k}\,c_k^{6}\right)\right)=\frac{1}{560}$$

$$\sum_{i=5}^{18} b_i\,c_i\left(\sum_{j=4}^{i-1} a_{i,j}\left(\sum_{k=3}^{j-1} a_{j,k}\left(\sum_{l=2}^{k-1} a_{k,l}\,c_l^{5}\right)\right)\right)=\frac{1}{3360}$$

$$\sum_{i=6}^{18} b_i\,c_i\left(\sum_{j=5}^{i-1} a_{i,j}\left(\sum_{k=4}^{j-1} a_{j,k}\left(\sum_{l=3}^{k-1} a_{k,l}\left(\sum_{m=2}^{l-1} a_{l,m}\,c_m^{4}\right)\right)\right)\right)=\frac{1}{16800}$$

can be solved to give values for $a_{15,10},\ a_{15,11},\ a_{17,12},$ by using the mult-dimensional form of Newton's method provided with suitable initial approximations.
In practice it is often sufficient to give an an initial common value such as 0 or 0.1 for the three variables.

Step 7:
We turn our attention to the order 9 embedded scheme. We remove rows 15 to 18 from the order 10 scheme and add three new rows corresponding two three nodes $c_{15},\ c_{16}$ and $c_{17}=1$..
We specify the nodes c_{15} and c_{16} and also the linking coefficients $a_{17,10}$ and $a_{17,14}$.
We also specify that

$$b^*_i=0,\ \ i=2\,..\,8,$$

$$a_{15,j}=0,\ j=2\,..\,5,\quad a_{16,j}=0,\ j=2\,..\,5,\quad a_{17,j}=0,\ j=2\,..\,5.$$

The weights are not completely determined by the quadrature conditions alone and attempting to leave one of the weights as an unknown bleads to problems.
However a value for b^*_9 can be obtained from the requirement that a certain order 9 order condition be satisfied, namely:

$$\sum_{i=3}^{17} b^*_i\,c_i^{2}\left(\sum_{j=2}^{i-1} a_{i,j}\,c_j^{5}\right)=\frac{1}{54}.$$

This can be achieved by constructing two "pseudo" schemes for nearby values of b^*_9 , that is, coefficient systems that only fail to satisfy the order conditions for an order 9 scheme by virtue of the fact that the previous mentioned order condition is not satsfied. Then the value that ensures that the order condition is satisfied can be found by linearity.
We assume that an accurate value for b^*_9 has been found and indicate how the construction of the order 9 scheme procedes.
We construct a system of equations from the order 9 quadrature conditions

$$\sum_{i=1}^{17} b^*_i=1,\qquad \sum_{i=1}^{17} b^*_i\,c_i^{k}=\frac{1}{k+1}\ \text{ for }\ k=1\,..\,8.$$

This enables us to solve this system of equations for the weights b^*_1 and b^*_{10} to b^*_{17} of the embedded scheme.

Step 8:
The specify that the stages 15, 16 and 17 are to have stage-order 6, 5 and 5 respectively, so tht the following table of stage-orders will apply to the embedded scheme.

stage	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
stage-order	2	3	3	4	4	5	5	5	5	6	6	6	6	5	5

Hence the following conditions hold.

$$\sum_{j=1}^{14} a_{15,j} = c_{15}, \qquad \sum_{j=2}^{14} a_{15,j}\, c_j^{(k-1)} = \frac{1}{k} c_{15}^{\;k}, \quad k = 2\,..\,6.$$

$$\sum_{j=1}^{15} a_{16,j} = c_{16}, \qquad \sum_{j=2}^{15} a_{16,j}\, c_j^{(k-1)} = \frac{1}{k} c_{16}^{\;k}, \quad k = 2\,..\,5.$$

$$\sum_{j=1}^{16} a_{17,j} = c_{17}, \qquad \sum_{j=2}^{16} a_{17,j}\, c_j^{(k-1)} = \frac{1}{k} c_{17}^{\;k}, \quad k = 2\,..\,5.$$

We can obtain expressions for the 16 linking coefficients

$$a_{15,6},\ a_{15,7},\ a_{15,9},\ a_{15,11},\ a_{15,12},\ a_{15,13},\ a_{16,6},\ a_{16,8},\ a_{16,9},\ a_{16,12},\ a_{16,14},$$
$$a_{17,1},\ a_{17,7},\ a_{17,11},\ a_{17,13},\ a_{17,15},$$

in terms of the other linking coefficients.
At this stage the following 15 linking coefficients are unknown.

$$a_{15,1},\ a_{15,8},\ a_{15,10},\ a_{15,14},\ a_{16,1},\ a_{16,7},\ a_{16,10},\ a_{16,11},\ a_{16,13},\ a_{16,15},$$
$$a_{17,6},\ a_{17,8},\ a_{17,9},\ a_{17,12},\ a_{17,16}.$$

Step 9:
We now construct a system of equations from the column simplfying conditions

$$\sum_{i=2}^{15} b^*_i\, a_{i,1} = b^*_1, \qquad \sum_{i=j+1}^{15} b^*_i\, a_{i,j} = b^*_j\,(1 - c_j), \quad j = 6, 7, 8, 10, 15, 16.$$

We can solve this system to obtain expressions for the 7 linking coefficients

$$a_{15,8},\ a_{16,1},\ a_{16,7},\ a_{16,10},\ a_{16,11},\ a_{16,15},\ a_{17,16},$$

in terms of the other linking coefficients.
By means of substitution we obtain linear expressions for 23 linking coefficients with the following 8 linking coefficients remaining as parameters.

$$a_{15,1},\ a_{15,10},\ a_{15,14},\ a_{16,13},\ a_{17,6},\ a_{17,8},\ a_{17,9},\ a_{17,12}.$$

Step 10:
We now start to consider order conditions starting with the following two order 8 order conditions.
In detail, these order conditions are:

$$\sum_{i=3}^{17} b^*_i\, c_i \left(\sum_{j=2}^{i-1} a_{i,j}\, c_j^{\;5}\right) = \frac{1}{48}, \qquad \sum_{i=4}^{17} b^*_i\, c_i \left(\sum_{j=3}^{i-1} a_{i,j} \left(\sum_{k=2}^{j-1} a_{j,k}\, c_k^{\;4}\right)\right) = \frac{1}{240}.$$

A system of equations arising from these two conditions can be solved to give expressions for the 2 linking coefficients

$$a_{16,13},\ a_{17,9},$$

in terms of the other linking coefficients.
By means of substitution we obtain linear expressions for 25 linking coefficients with the following 6 linking coefficients remaining as parameters.

$$a_{15,1},\ a_{15,10},\ a_{15,14},\ a_{17,6},\ a_{17,8},\ a_{17,12}.$$

Step 11:
The single equation given by the simple order condition

$$\sum_{i=5}^{17} b^*_i\, c_i \left(\sum_{j=4}^{i-1} a_{i,j} \left(\sum_{k=3}^{j-1} a_{j,k} \left(\sum_{l=2}^{k-1} a_{k,l}\, c_l^{\;3}\right)\right)\right) = \frac{1}{960}.$$

can be solved to give an expression for $a_{15,14}$.
By means of substitution we obtain linear expressions for 26 linking coefficients with the following 5 linking coefficients remaining as parameters.

$$a_{15,1},\ a_{15,10},\ a_{17,6},\ a_{17,8},\ a_{17,12}.$$

Step 12:
At this stage the coefficient satisfies all the order 8 order conditions so we start to consider order 9 order conditions starting with the condition

$$\sum_{i=4}^{17} b^*_i \, c_i^2 \left(\sum_{j=3}^{i-1} a_{i,j} \left(\sum_{k=2}^{j-1} a_{j,k} \, c_k^4 \right) \right) = \frac{1}{270}.$$

We can solve the equation given by this order condition to give an expression for $a_{17,\,12}$.
By means of substitution we obtain linear expressions for 27 linking coefficients with the following 4 linking coefficients remaining as parameters.

$$a_{15,\,1},\ a_{15,\,10},\ a_{17,\,6},\ a_{17,\,8}.$$

Step 13:
The single equation given by the simple order condition

$$\sum_{i=3}^{17} b^*_i \, c_i \left(\sum_{j=2}^{i-1} a_{i,j} \, c_j^6 \right) = \frac{1}{63}$$

can be solved to give an expression for $a_{17,\,6}$.
By means of substitution we obtain linear expressions for 28 linking coefficients with the following 3 linking coefficients remaining as parameters.

$$a_{15,\,1},\ a_{15,\,10},\ a_{17,\,8}.$$

Step 14:
The single equation given by the simple order condition

$$\sum_{i=3}^{17} b^*_i \, c_i \left(\sum_{j=2}^{i-1} a_{i,j} \left(\sum_{k=1}^{j-1} a_{j,k} \, c_k^5 \right) \right) = \frac{1}{378}$$

can be solved to give an expression for $a_{15,\,10}$.
By means of substitution we obtain linear expressions for 29 linking coefficients with the following 2 linking coefficients remaining as parameters.

$$a_{15,\,1},\ a_{17,\,8}.$$

Step 15:
We can solve the system of equations given by the two order conditions

$$\sum_{i=5}^{17} b^*_i \, c_i \left(\sum_{j=4}^{i-1} a_{i,j} \left(\sum_{k=3}^{j-1} a_{j,k} \left(\sum_{l=2}^{k-1} a_{k,l} \, c_l^4 \right) \right) \right) = \frac{1}{1890},$$

$$\sum_{i=6}^{17} b^*_i \, c_i \left(\sum_{j=5}^{i-1} a_{i,j} \left(\sum_{k=4}^{j-1} a_{j,k} \left(\sum_{l=3}^{k-1} a_{k,l} \left(\sum_{m=2}^{l-1} a_{l,m} \, c_m^3 \right) \right) \right) \right) = \frac{1}{7560}$$

to obtain values for $a_{15,\,1}$ and $a_{17,\,8}$.
The two resulting equations have degree 2 in the two variables but the system actually reduces to a linear system.
Once values for these two coefficients are found we can obtain values for all the 31 linking coefficients.

The order 9 scheme can be embedded in the order 10 scheme by inserting the 15th, 16th and 17th rows as 19th, 20th and 21st rows of a combined scheme respectively (with the some horizontal adjustments and addition of zero coefficients). In detail, the nodes c_{15}, c_{16} and c_{17} of the 17 stage order 9 scheme become the nodes c_{19}, c_{20} and c_{21} of the combined scheme respectively, and the first 14 entries in the 15th, 16th and 17th rows of the order 9 scheme are inserted as the first 14 entries of rows 19, 20 and 21 of the combined scheme respectively. The linking coefficients $a_{19,\,15}$ to $a_{19,\,18}$ are zero as are $a_{20,\,15}$ to $a_{20,\,18}$ and $a_{21,\,15}$ to $a_{21,\,18}$. The linking coefficients $a_{20,\,19}$ of the combined scheme is $a_{16,\,15}$ of the 17 stage order 9 scheme. The linking coefficients $a_{21,\,19}$ and $a_{21,\,20}$ of the combined scheme are the linking coefficients $a_{17,\,15}$ and $a_{17,\,16}$ respectively of the 17 stage order 9 scheme. Similarly, the weights b^*_1 to b^*_{14} of the order 9 scheme become the first 14 weights of the embedded scheme and the weights b^*_{15} to b^*_{18} are zero. The weights b^*_{19}, b^*_{20} and b^*_{21} of the combined scheme are the weights b^*_{15}, b^*_{16} and b^*_{17} of the 17 stage order 9 scheme respectively. We can set $b_{19} = 0$, $b_{20} = 0$ and $b_{21} = 0$ to make the order 10 scheme into a 21 stage scheme.

The principal error of an order 10 scheme constructed in the manner described above can be calculated using 25 of the 1842 principal error terms. These error terms are given in an abreviated form as follows.

$$b\,c\;(a\;(a\;(a\;(a\;(a\;(a\;(a\;(a\,c))))))))-\frac{1}{3991680},\qquad b\,c\;(a\,c\;(a\;(a\;(a\;(a\;(a\;(a\,c)))))))-\frac{1}{498960},$$

$$\frac{1}{2}\left(b\,c\;(a\;(a\;(a\;(a\;(a\;(a\;(a\,c^2)))))))-\frac{1}{1995840}\right),\qquad \frac{1}{2}\left(b\,c^2\;(a\;(a\;(a\;(a\;(a\;(a\;(a\,c)))))))-\frac{1}{443520}\right),$$

$$\frac{1}{2}\left(b\,c\;(a\,c\;(a\;(a\;(a\;(a\;(a\,c^2))))))-\frac{1}{249480}\right),\qquad \frac{1}{4}\left(b\,c^2\;(a\;(a\;(a\;(a\;(a\;(a\,c^2))))))-\frac{1}{221760}\right),$$

$$\frac{1}{6}\left(b\,c\;(a\;(a\;(a\;(a\;(a\;(a\,c^3))))))-\frac{1}{665280}\right),\qquad \frac{1}{6}\left(b\,c^3\;(a\;(a\;(a\;(a\;(a\;(a\,c))))))-\frac{1}{55440}\right),$$

$$\frac{1}{6}\left(b\,c\;(a\,c\;(a\;(a\;(a\;(a\,c^3)))))-\frac{1}{83160}\right),\qquad \frac{1}{12}\left(b\,c^2\;(a\;(a\;(a\;(a\;(a\,c^3)))))-\frac{1}{73920}\right),$$

$$\frac{1}{12}\left(b\,c^3\;(a\;(a\;(a\;(a\;(a\,c^2)))))-\frac{1}{27720}\right),\qquad \frac{1}{24}\left(b\,c\;(a\;(a\;(a\;(a\;(a\,c^4)))))-\frac{1}{166320}\right),$$

$$\frac{1}{36}\left(b\,c^3\;(a\;(a\;(a\;(a\,c^3))))-\frac{1}{9240}\right),\qquad \frac{1}{24}\left(b\,c\;(a\,c\;(a\;(a\;(a\,c^4))))-\frac{1}{20790}\right),$$

$$\frac{1}{48}\left(b\,c^2\;(a\;(a\;(a\;(a\,c^4))))-\frac{1}{18480}\right),\qquad \frac{1}{120}\left(b\,c\;(a\;(a\;(a\;(a\,c^5))))-\frac{1}{33264}\right),$$

$$\frac{1}{144}\left(b\,c^3\;(a\;(a\;(a\,c^4)))-\frac{1}{2310}\right),\qquad \frac{1}{120}\left(b\,c\;(a\,c\;(a\;(a\,c^5)))-\frac{1}{4158}\right),\qquad \frac{1}{240}\left(b\,c^2\;(a\;(a\;(a\,c^5)))-\frac{1}{3696}\right),$$

$$\frac{1}{720}\left(b\,c\;(a\;(a\;(a\,c^6)))-\frac{1}{5544}\right),\qquad \frac{1}{720}\left(b\,c^3\;(a\;(a\,c^5))-\frac{1}{462}\right),\qquad \frac{1}{720}\left(b\,c\;(a\,c\;(a\,c^6))-\frac{1}{693}\right),$$

$$\frac{1}{1440}\left(b\,c^2\;(a\;(a\,c^6))-\frac{1}{616}\right),\qquad \frac{1}{4320}\left(b\,c^3\;(a\,c^6)-\frac{1}{77}\right),\qquad \frac{1}{3628800}\left(b\,c^{10}-\frac{1}{11}\right).$$

For example, $\frac{1}{240}\left(b\,c^2\;(a\;(a\;(a\,c^5)))-\frac{1}{3696}\right)$ is an abreviation for

$$\frac{1}{240}\left(\left(\sum_{i=2}^{21} b_i\,c_i^2\left(\sum_{j=3}^{i-1} a_{i,j}\left(\sum_{k=4}^{j-1} a_{j,k}\left(\sum_{l=5}^{k-1} a_{k,l}\,c_l^5\right)\right)\right)\right)-\frac{1}{3696}\right).$$

The principal error norm can be calculated as $\sqrt{\sum_{i=1}^{25} d_i\,e_i^2}$, where e_i is the value of the ith error term above and d_i is the ith member of the sequence

2, 2, 2, 6, 2, 6, 20, 30, 20, 60, 30, 156, 300, 156, 468, 2052, 2340, 2052, 6156, 39432, 30780, 39432, 118296, 591480, 173908721112.

The principal error norm of the associated order 9 embedded scheme can be calculated using 24 of the 719 principal error terms. These error terms are given in an abreviated form as follows.

$$b^*\,c\;(a\;(a\;(a\;(a\;(a\;(a\;(a\,c)))))))-\frac{1}{403200},\qquad b^*\,c\;(a\,c\;(a\;(a\;(a\;(a\;(a\,c))))))-\frac{1}{57600},$$

$$\frac{1}{2}\left(b^*\,c\;(a\;(a\;(a\;(a\;(a\;(a\,c^2))))))-\frac{1}{201600}\right),\qquad \frac{1}{2}\left(b^*\,c^2\;(a\;(a\;(a\;(a\;(a\;(a\,c))))))-\frac{1}{50400}\right),$$

$$\frac{1}{2}\left(b* c\ (a\,c\ (a\ (a\ (a\ (a\,c^2)))))-\frac{1}{28800}\right),\quad \frac{1}{4}\left(b* c^2\ (a\ (a\ (a\ (a\ (a\,c^2)))))-\frac{1}{25200}\right),$$

$$\frac{1}{6}\left(b* c\ (a\ (a\ (a\ (a\ (a\,c^3)))))-\frac{1}{67200}\right),\quad \frac{1}{6}\left(b* c^3\ (a\ (a\ (a\ (a\ (a\,c)))))-\frac{1}{7200}\right),$$

$$\frac{1}{6}\left(b* c\ (a\,c\ (a\ (a\ (a\,c^3))))-\frac{1}{9600}\right),\quad \frac{1}{12}\left(b* c^2\ (a\ (a\ (a\ (a\,c^3))))-\frac{1}{8400}\right),$$

$$\frac{1}{12}\left(b* c^3\ (a\ (a\ (a\ (a\,c^2))))-\frac{1}{3600}\right),\quad \frac{1}{24}\left(b* c\ (a\ (a\ (a\ (a\,c^4))))-\frac{1}{16800}\right),$$

$$\frac{1}{36}\left(b* c^3\ (a\ (a\ (a\,c^3)))-\frac{1}{1200}\right),\quad \frac{1}{24}\left(b* c\ (a\,c\ (a\ (a\,c^4)))-\frac{1}{2400}\right),\quad \frac{1}{48}\left(b* c^2\ (a\ (a\ (c^4\,a)))-\frac{1}{2100}\right),$$

$$\frac{1}{120}\left(b* c\ (a\ (a\ (a\,c^5)))-\frac{1}{3360}\right),\quad \frac{1}{144}\left(b* c^3\ (a\ (c^4\,a))-\frac{1}{300}\right),\quad \frac{1}{120}\left(b* c\ (a\,c\ (a\,c^5))-\frac{1}{480}\right),$$

$$\frac{1}{240}\left(b* c^2\ (a\ (a\,c^5))-\frac{1}{420}\right),\quad \frac{1}{720}\left(b* c\ (a\ (a\,c^6))-\frac{1}{560}\right),\quad \frac{1}{720}\left(b* c^3\ (a\,c^5)-\frac{1}{60}\right),$$

$$\frac{1}{1440}\left(b* c^2\ (a\,c^6)-\frac{1}{70}\right),\quad \frac{1}{5040}\left(b* c\ (a\,c^7)-\frac{1}{80}\right),\quad \frac{1}{362880}\left(b* c^9-\frac{1}{10}\right).$$

For example, $\frac{1}{120}\left(b* c\ (a\ (a\ (a\,c^5)))-\frac{1}{3360}\right)$ is an abreviation for

$$\frac{1}{120}\left(\left(\sum_{i=2}^{17} b*_i\,c_i\left(\sum_{j=3}^{i-1} a_{i,j}\left(\sum_{k=4}^{j-1} a_{j,k}\left(\sum_{l=5}^{k-1} a_{k,l}\,c_l^{\,5}\right)\right)\right)\right)-\frac{1}{3360}\right).$$

The principal error norm can be calculated as $\sqrt{\sum_{i=1}^{24} d_i\,e_i^{\,2}}$, where e_i is the value of the ith error term above and d_i is the ith member of the sequence

2, 2, 2, 6, 2, 6, 20, 30, 20, 60, 30, 156, 300, 156, 468, 2052, 2340, 2052, 6156, 39432, 30780, 118296, 840744, 1295337252.

The following sets of parameter values determine schemes with the given characteristics. A complete list of the values of the coefficients (correct to 85 digits) for the 3 schemes listed here are given in separate documents.

1. **Scheme for which the principal error norm is close to a minimum**.

The 18 stage order 10 scheme is determined by the parameter values

$$c_2=\frac{709}{17315},\ c_5=\frac{40698}{72565},\ c_{10}=\frac{10511}{90543},\ c_{11}=\frac{19869}{51119},\ a_{11,\,10}=\frac{7621}{223149},\ a_{14,\,10}=-\frac{3095}{1709},$$

while the 17 stage order 9 scheme is determined by the parameter values

$$c_{15}=\frac{5}{23},\ c_{16}=\frac{21}{25},\ a_{17,\,10}=\frac{3}{10},\ a_{17,\,14}=-\frac{2}{27}.$$

After renaming when the new stages are include after the 18 stages of the order 10 scheme these become

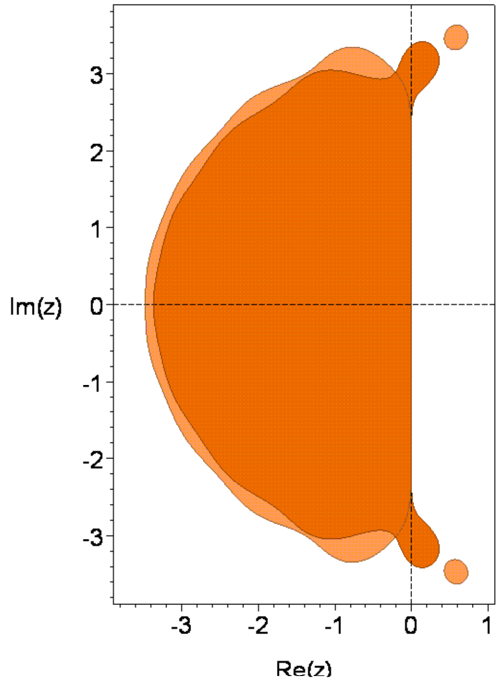
$$c_{19}=\frac{5}{23},\ c_{20}=\frac{21}{25},\ a_{21,\,10}=\frac{3}{10},\ a_{21,\,14}=-\frac{2}{27}.$$

The principal error norm of the order 10 scheme is $0.2497052077 \times 10^{(-6)}$ and the real stability interval is $[-3.47701, 0]$.

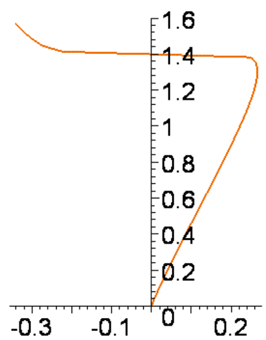
The principal error norm of the order 9 scheme is $0.1083857090 \times 10^{(-4)}$ and the real stability interval is $[-3.3707, 0]$.

The maximum magnitude of the linking coefficients is 9.030534255 and the 2-norm of the linking coefficients is 23.96732424.

The stability regions of the two schemes are shown in the following picture in which the stability region of the order 9 scheme is given the darker shade.



The following picture shows the result of distorting the boundary curve of the stability region of the order 10 scheme horizontally by taking the 11th root of the real part of points along the curve.



The stability region intersects the nonnegative imaginary axis in the interval $[0, 1.40799\, i]$.

2. **Scheme with a moderately large stability region**.

The 18 stage order 10 scheme is determined by the parameter values

$$c_2 = \frac{13}{518},\ c_5 = \frac{175}{319},\ c_{10} = \frac{97}{824},\ c_{11} = \frac{45}{116},\ a_{11,10} = \frac{41}{964},\ a_{14,10} = -\frac{389}{218},$$

while the 17 stage order 9 scheme is determined by the parameter values

$$c_{15} = \frac{22}{95},\ c_{16} = \frac{4}{5},\ a_{17,10} = \frac{55}{17},\ a_{17,14} = \frac{1}{5}.$$

After renaming when the new stages are include after the 18 stages of the order 10 scheme these become

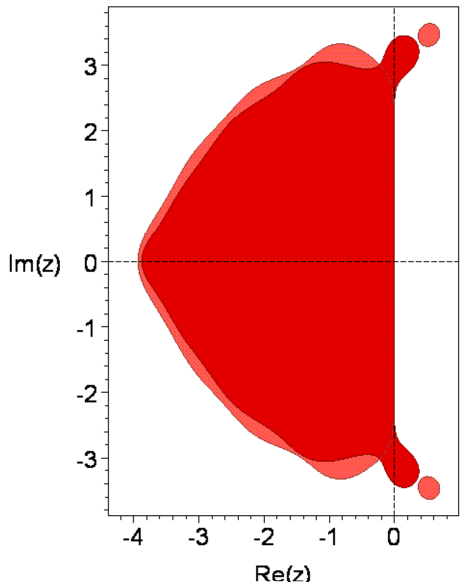
$$c_{19} = \frac{22}{95},\ c_{20} = \frac{4}{5},\ a_{21,10} = \frac{55}{17},\ a_{21,14} = \frac{1}{5}.$$

The principal error norm of the order 10 scheme is $0.2797129535 \times 10^{(-6)}$ and the real stability interval is $[-3.93592, 0]$.

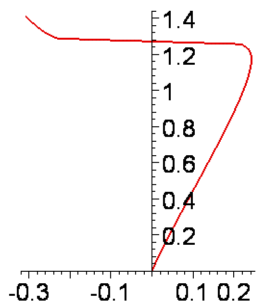
The principal error norm of the order 9 scheme is $0.1228271247 \times 10^{(-4)}$ and the real stability interval is $[-3.87594, 0]$.

The maximum magnitude of the linking coefficients is 9.251611659 and the 2-norm of the linking coefficients is 23.4045906007.

The stability regions of the two schemes are shown in the following picture in which the stability region of the order 9 scheme is given the darker shade.



The following picture shows the result of distorting the boundary curve of the stability region of the order 10 scheme horizontally by taking the 11th root of the real part of points along the curve.



The stability region intersects the nonnegative imaginary axis in the interval $[0, 1.2703\,i]$.

It is possible to construct order 10 schemes for which 68 of the 1842 principal error terms are zero, that is, such that certain of the order 11 order conditions are satisfied. In the construction of such schemes the linking coefficient $a_{11,\,10}$ is kept as a parameter so that on reaching the last step in the construction of the order 10 scheme there are five linking coefficient parameters

$$a_{11,\,10},\ a_{14,\,10},\ a_{15,\,10},\ a_{15,\,11},\ a_{17,\,12}.$$

The two order 11 order conditions

$$\sum_{i=10}^{18} b_i\, c_i \left(\sum_{j=9}^{i-1} a_{i,j} \left(\sum_{k=8}^{j-1} a_{j,k} \left(\sum_{l=7}^{k-1} a_{k,l} \left(\sum_{m=6}^{l-1} a_{l,m} \left(\sum_{n=5}^{m-1} a_{m,n} \left(\sum_{p=4}^{n-1} a_{n,p} \left(\sum_{q=3}^{p-1} a_{p,q} \left(\sum_{r=2}^{q-1} a_{q,r}\, c_r\right)\right)\right)\right)\right)\right)\right)\right) = \frac{1}{3991680},$$

$$\sum_{i=4}^{18} b_i\, c_i^{3} \left(\sum_{j=3}^{i-1} a_{i,j} \left(\sum_{k=2}^{j-1} a_{j,k}\, c_k^{5}\right)\right) = \frac{1}{462}$$

are included along with the three order 10 order conditions used in the previous construction method to form a system of five equations for the five unknown coefficients. These equations can then be solved by the multi-dimensional form of Newton's method.

3. **Scheme with 68 zero error terms**.

The 18 stage order 10 scheme is determined by the parameter values

$$c_2 = \frac{124}{743},\ c_5 = \frac{144}{991},\ c_{10} = \frac{1032}{8855},\ c_{11} = \frac{2467}{6364}.$$

The 17 stage order 9 scheme is determined by the parameter values

$$c_{15} = \frac{14}{55},\ c_{16} = \frac{5}{6},\ a_{17,\,10} = 5,\ a_{17,\,14} = -\frac{1}{4}.$$

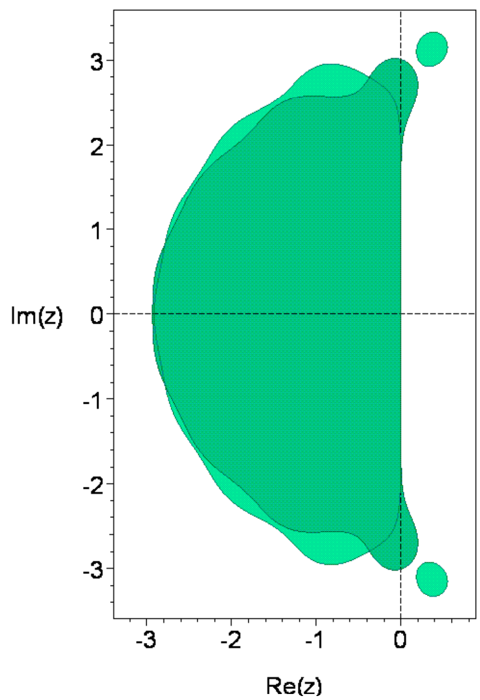
After renaming when the new stages are include after the 18 stages of the order 10 scheme these become

$$c_{19} = \frac{14}{55},\ c_{20} = \frac{5}{6},\ a_{21,\,10} = 5,\ a_{21,\,14} = -\frac{1}{4}.$$

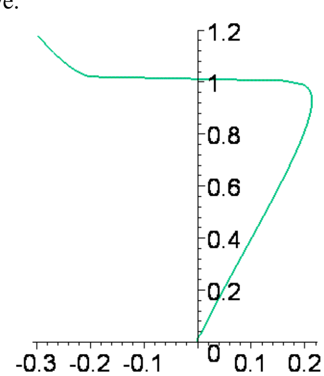
The principal error norm of the order 10 scheme is $0.2735765137 \times 10^{(-6)}$ and the real stability interval is $[-2.91452, 0]$.

The principal error norm of the order 9 scheme is $0.3131663792 \times 10^{(-4)}$ and the real stability interval is $[-2.93177, 0]$.
The maximum magnitude of the linking coefficients is 10.23323297 and the 2-norm of the linking coefficients is 25.99103221.

The stability regions of the two schemes are shown in the following picture in which the stability region of the order 9 scheme is given the darker shade.



The following picture shows the result of distorting the boundary curve of the stability region of the order 10 scheme horizontally by taking the 11th root of the real part of points along the curve.



The stability region intersects the nonnegative imaginary axis in the interval $[0, 1.00659\, i]$.

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